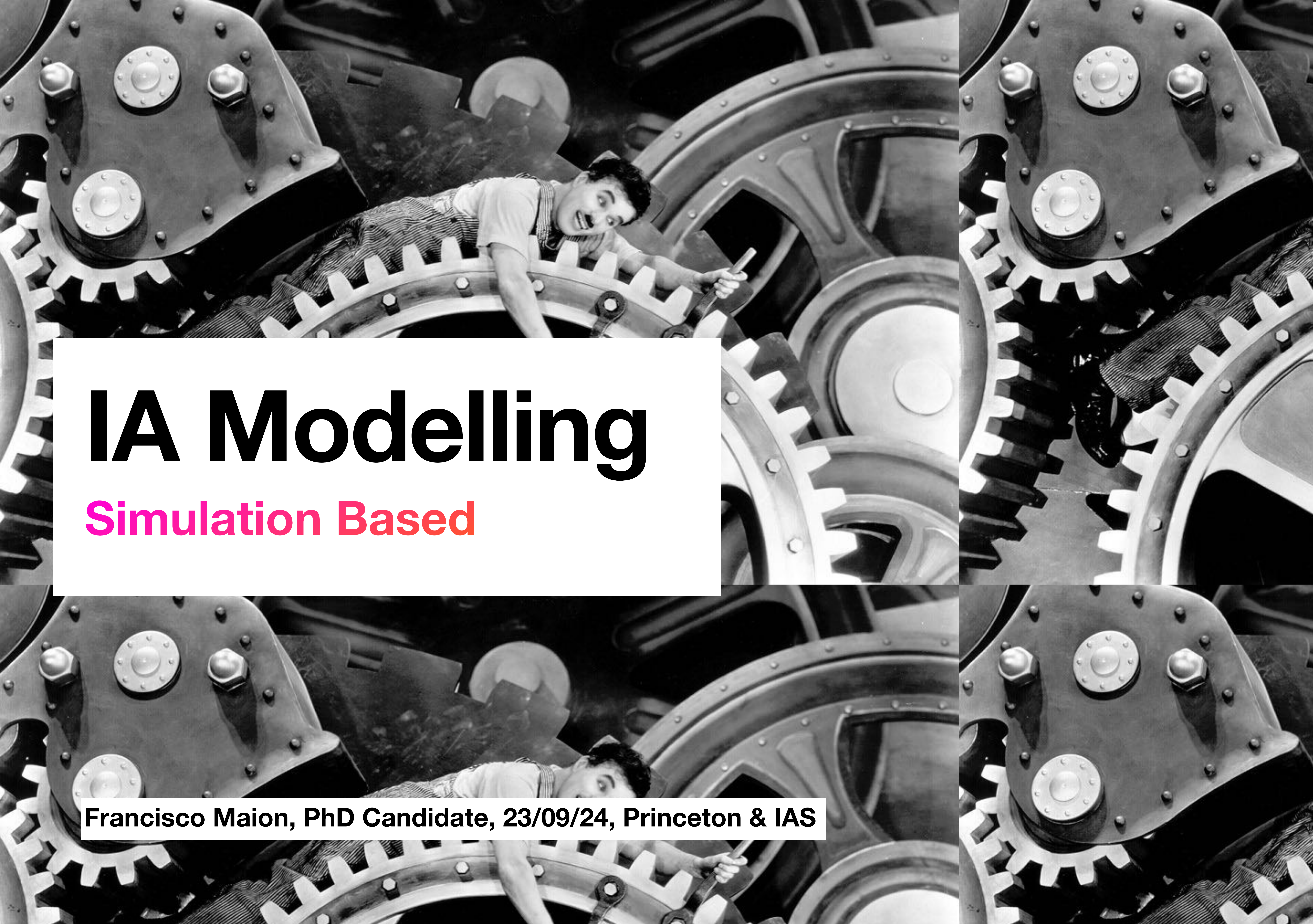


IA Modelling

Simulation Based

Francisco Maion, PhD Candidate, 23/09/24, Princeton & IAS

European Research Council

eman ta zabal zazu

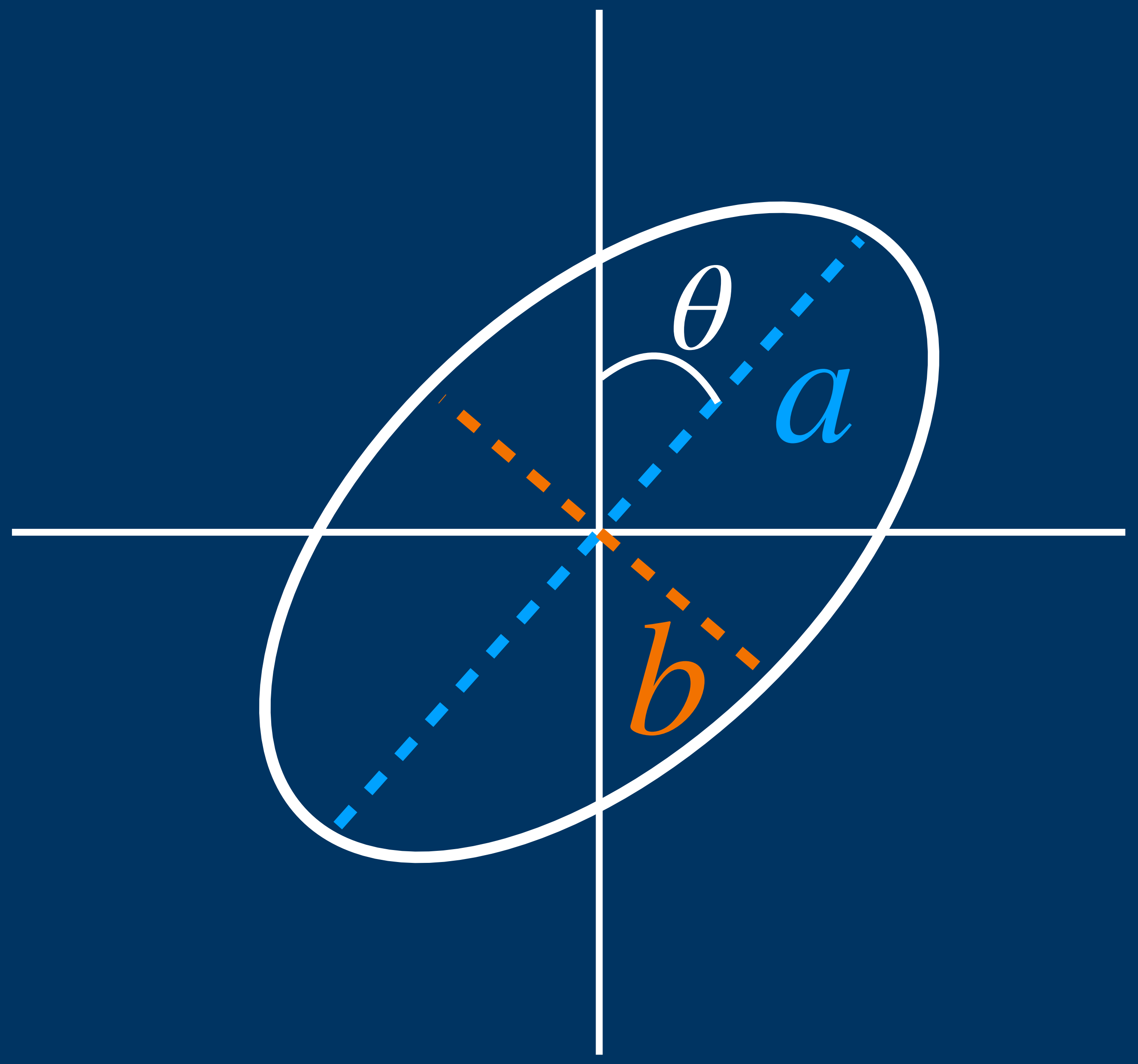
Universidad del País Vasco
Euskal Herriko Unibertsitatea

GOBIERNO DE ESPAÑA
MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

Introduction

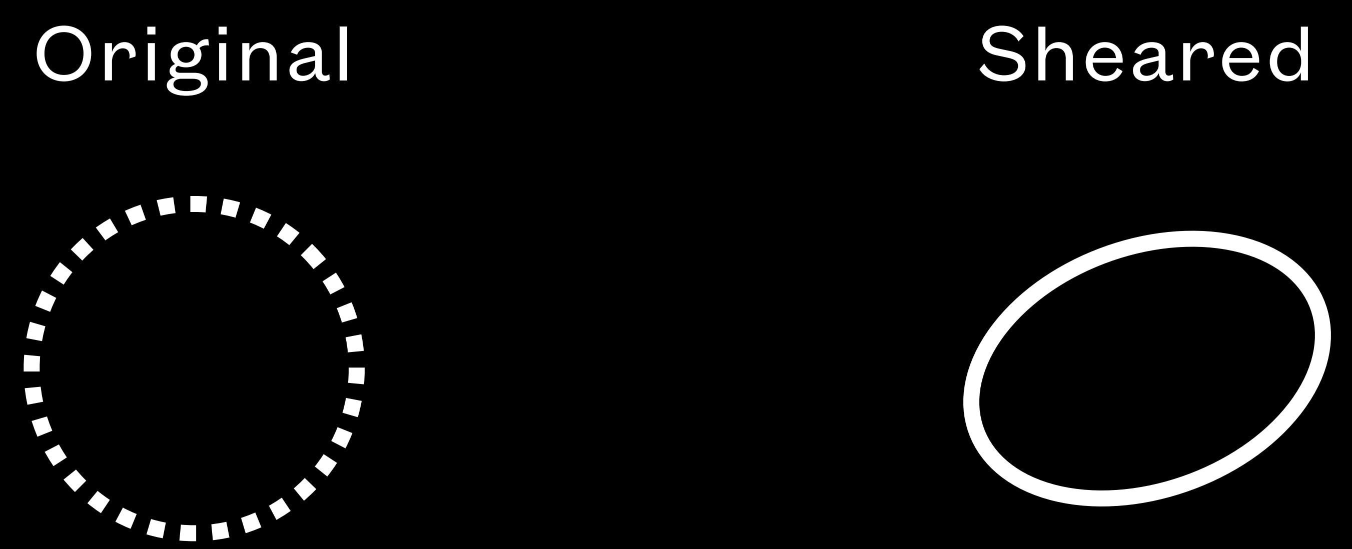
Cosmic-Shear

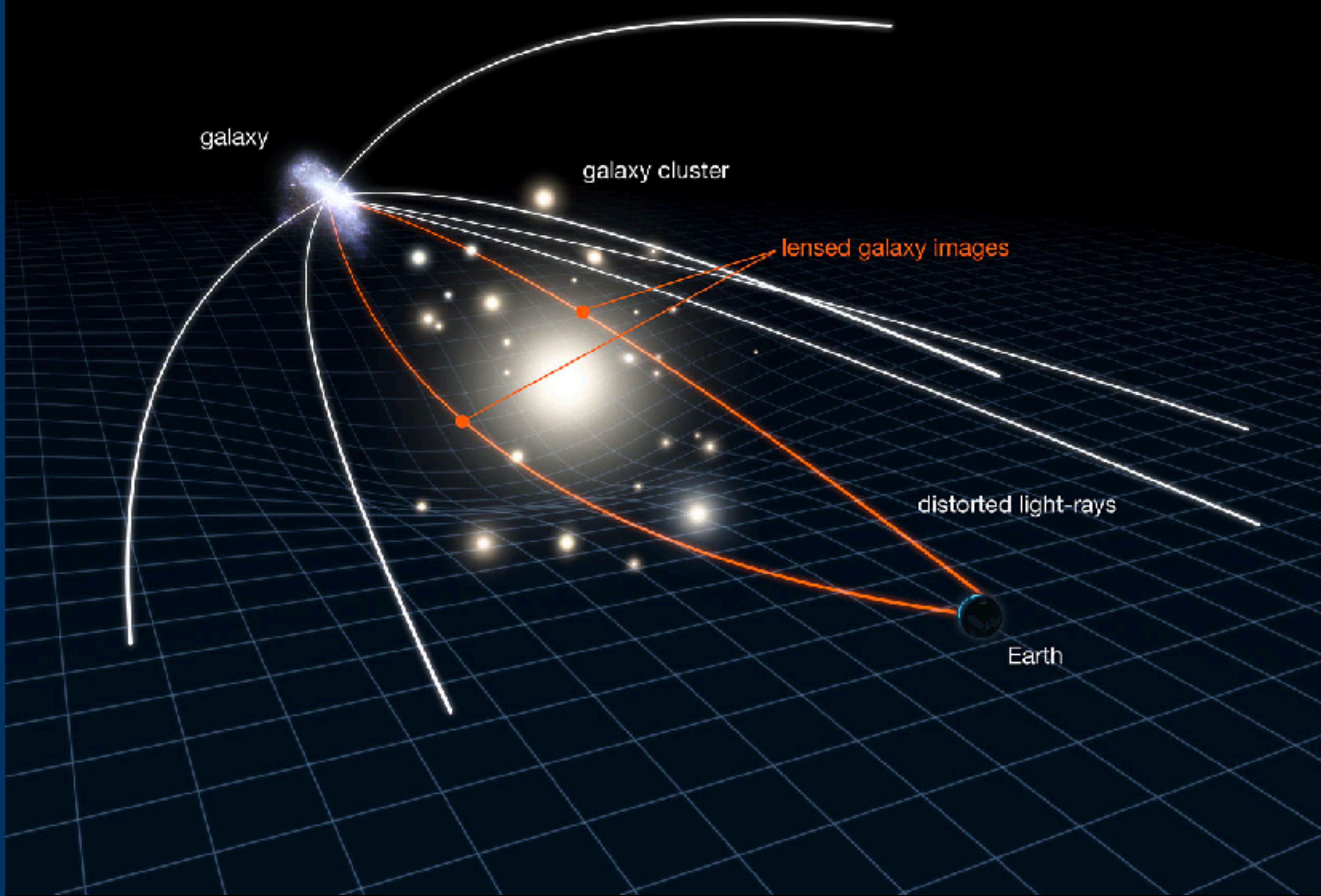
- Light travelling through the LSS gets gravitationally distorted
- Galaxy shapes will get distorted as well, or “sheared”



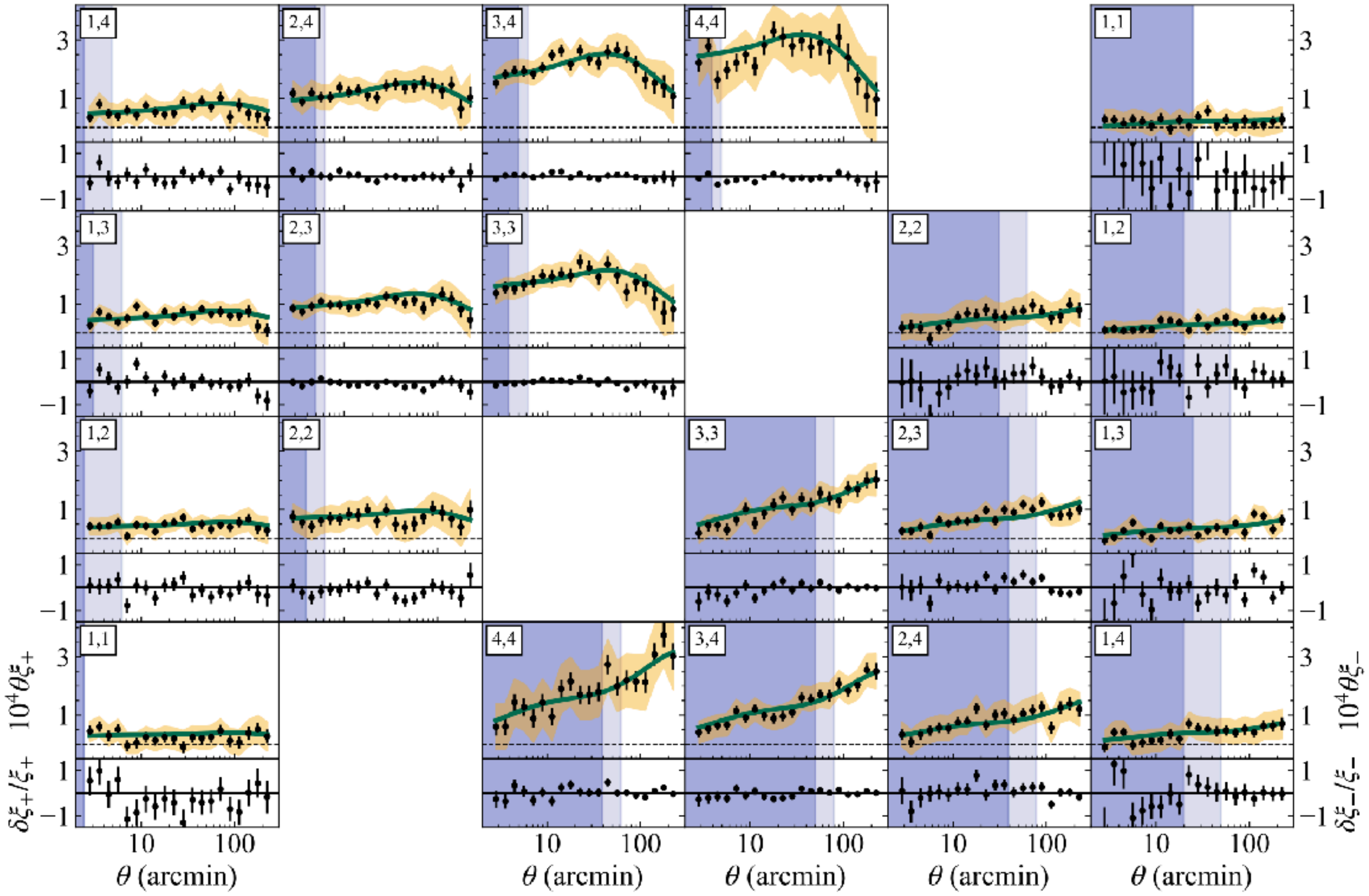
$$\varepsilon = \frac{a-b}{a+b} e^{2i\theta}$$

$$\varepsilon = \frac{\varepsilon^{(s)} + g}{1 + g^{*}\varepsilon^{(s)}} \approx \varepsilon^{(s)} + g$$





Cosmic-Shear



Amon et al (2021)

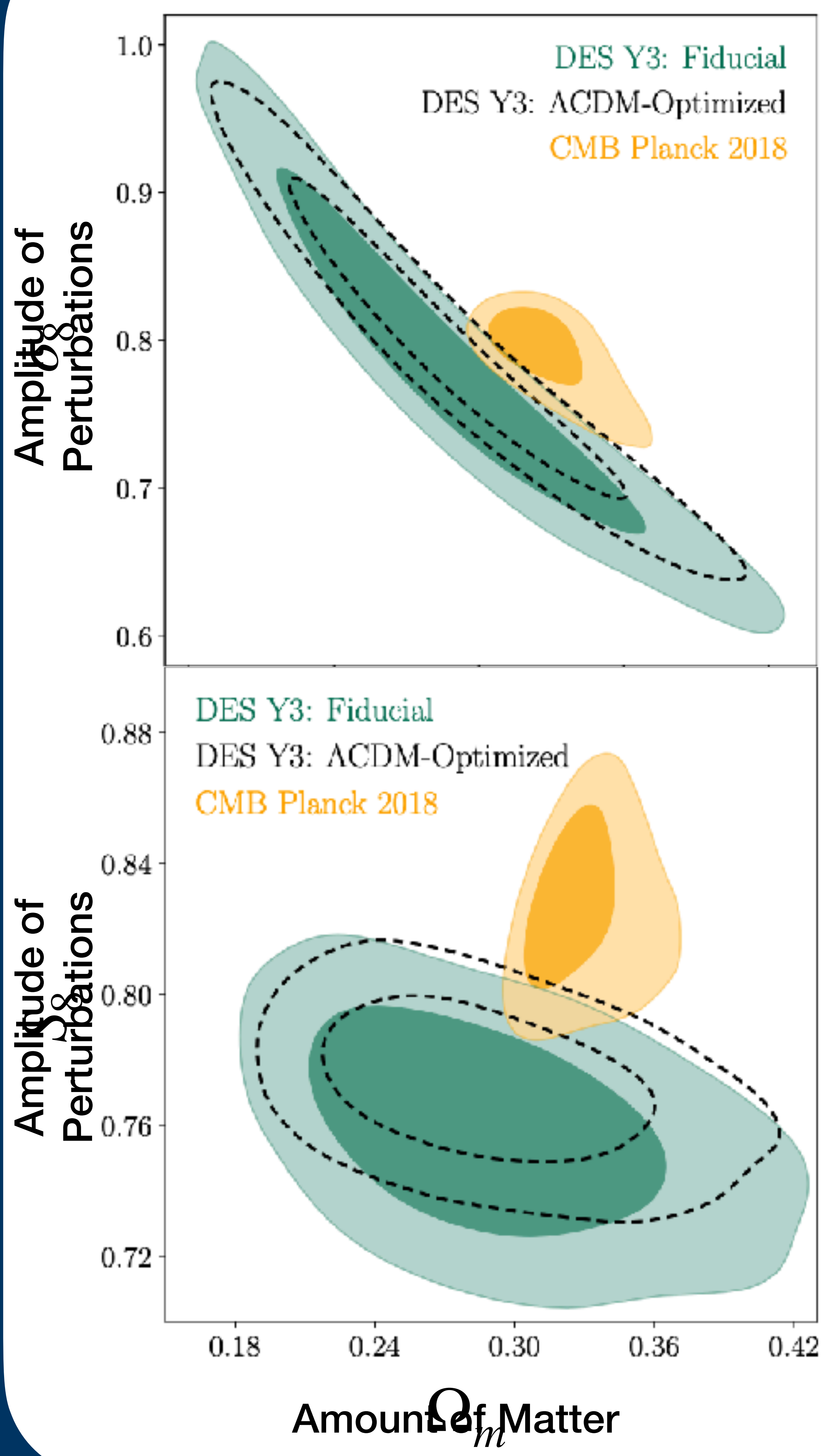
$$\xi_+^{ij} = \int_0^\infty \frac{d\ell\,\ell}{2\pi} J_0(\ell\theta) P_{ij}(\ell)$$

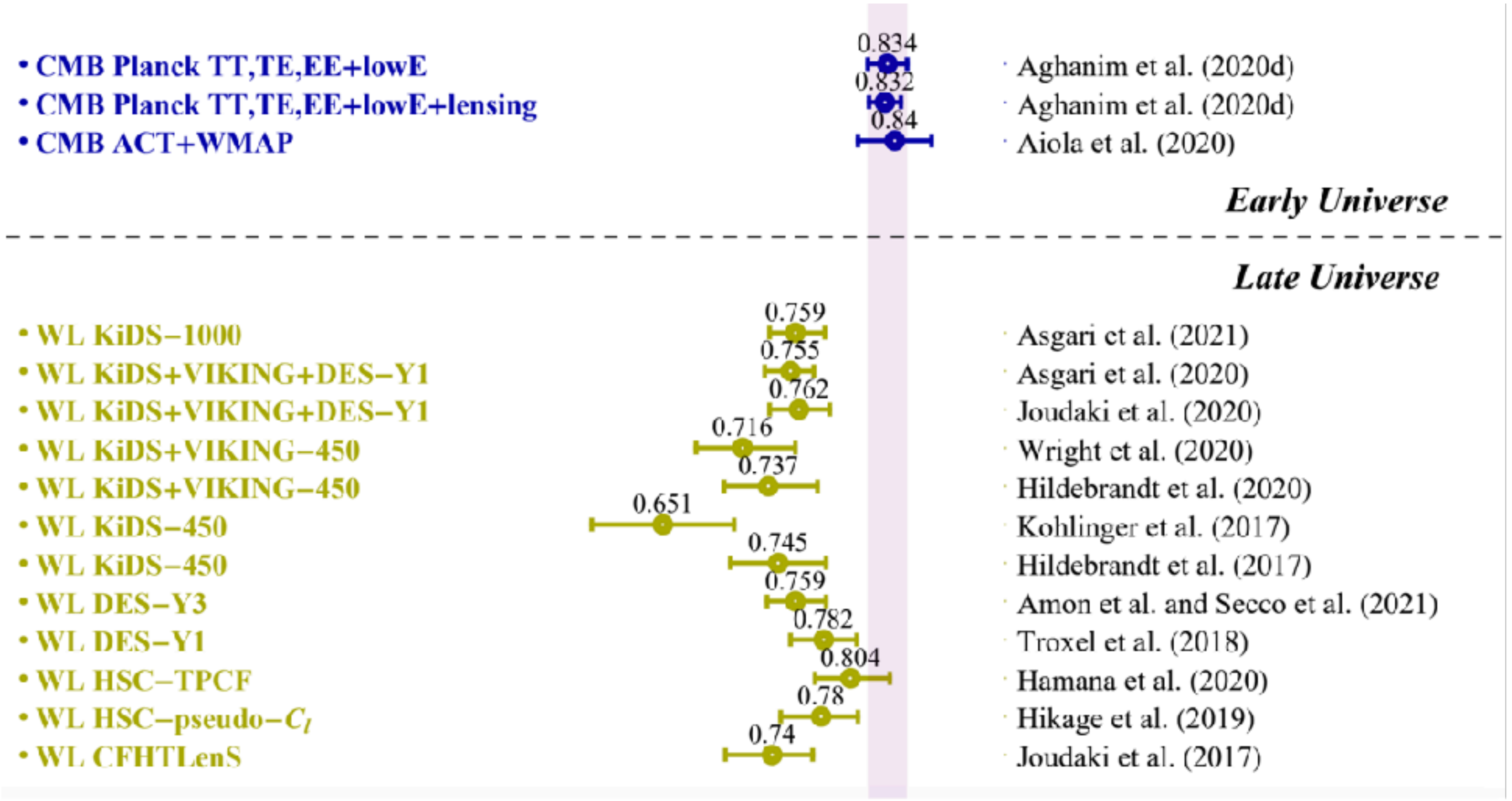
$$\xi_-^{ij} = \int_0^\infty \frac{d\ell\,\ell}{2\pi} J_4(\ell\theta) P_{ij}(\ell)$$

$$P_{ij}(\ell) = \int dw \frac{q_i(w) q_j(w)}{f_K^2(w)} P_\delta\left(\frac{\ell}{f_K(w)}, w\right)$$

DES-Y3

Adapted from
Amon et al (2021)
Secco & Samuroff (2021)

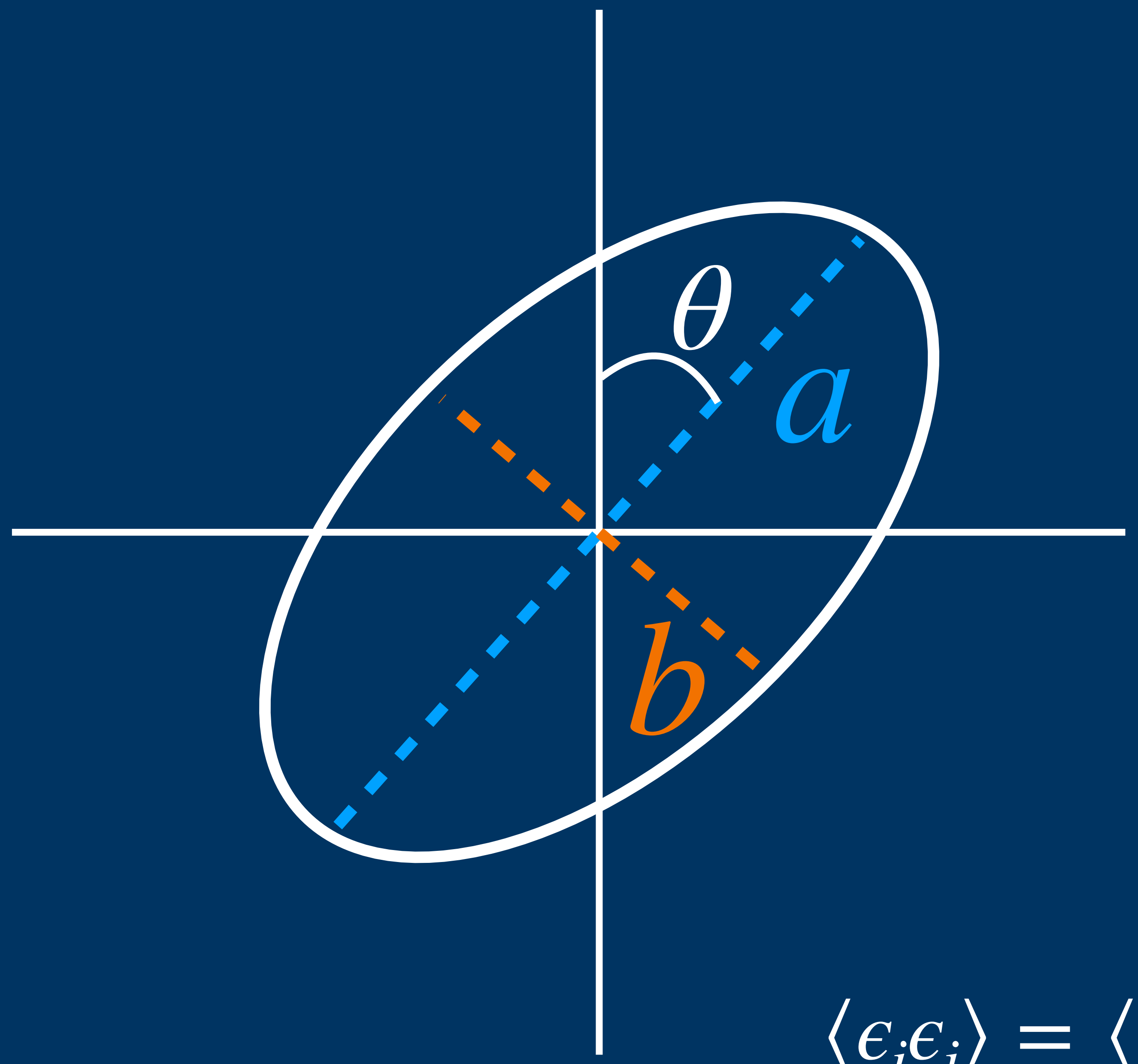




Adapted from “Cosmology Intertwined: A Review of the Particle Physics, Astrophysics, and Cosmology Associated with the Cosmological Tensions and Anomalies

Cosmic-Shear

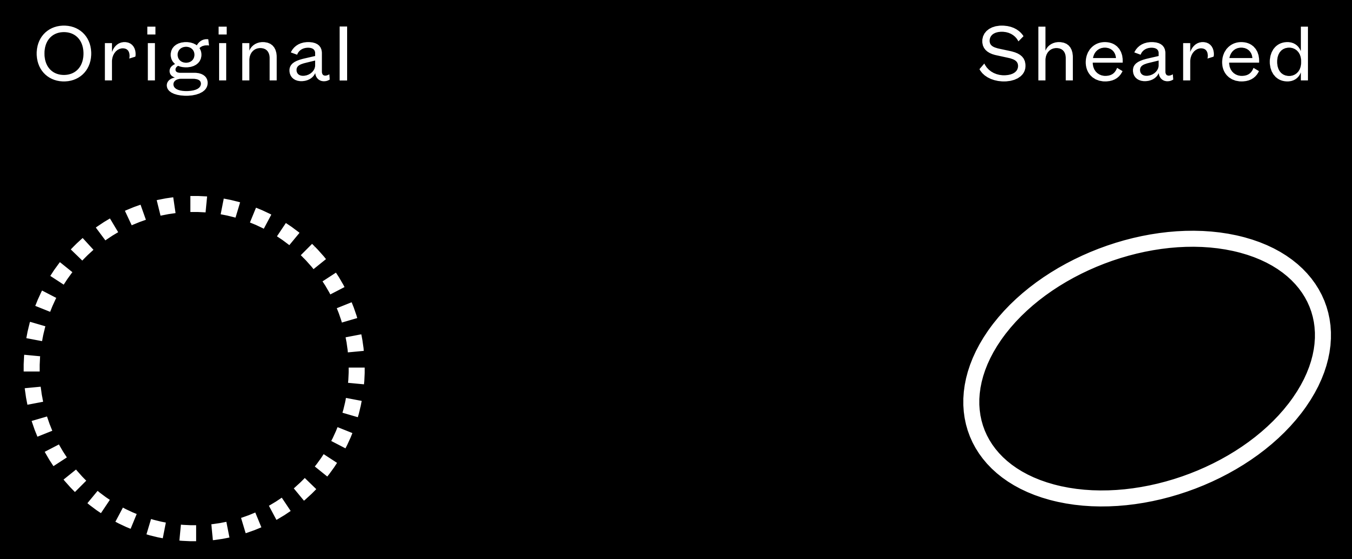
- Light travelling through the LSS gets gravitationally distorted
- Galaxy shapes will get distorted as well, or “sheared”

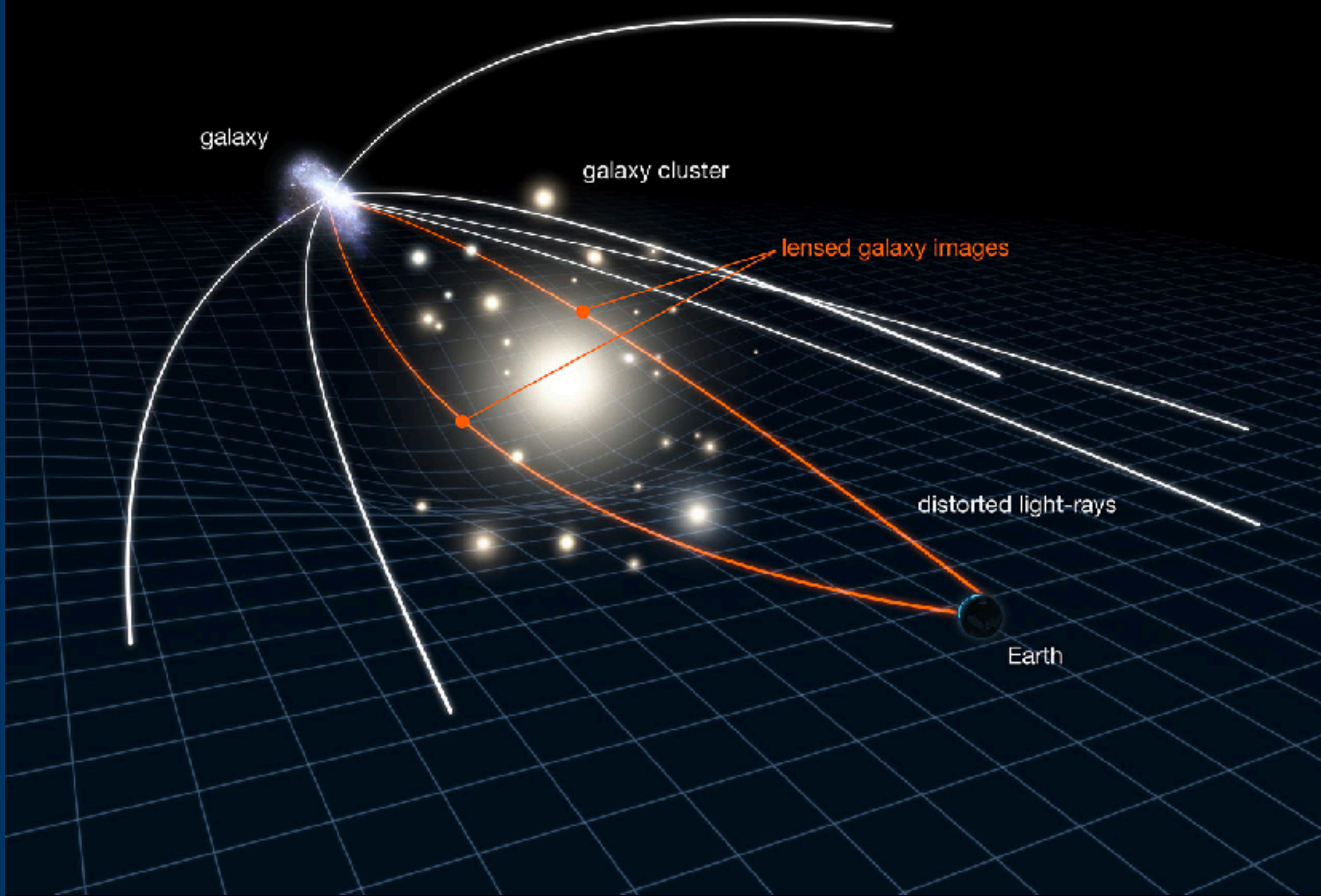


$$\varepsilon = \frac{a-b}{a+b} e^{2i\theta}$$

$$\varepsilon = \frac{\varepsilon^{(s)} + g}{1 + g^{*}\varepsilon^{(s)}} \approx \varepsilon^{(s)} + g$$

$$\langle \epsilon_i \epsilon_j \rangle = \underbrace{\langle g_i g_j \rangle}_{GG} + \underbrace{\langle \epsilon_i^{(s)} \epsilon_j^{(s)} \rangle}_{II} + \underbrace{\langle \epsilon_i^{(s)} g_j \rangle}_{IG} + \underbrace{\langle g_i \epsilon_j^{(s)} \rangle}_{GI} .$$





Intrinsic-Alignments

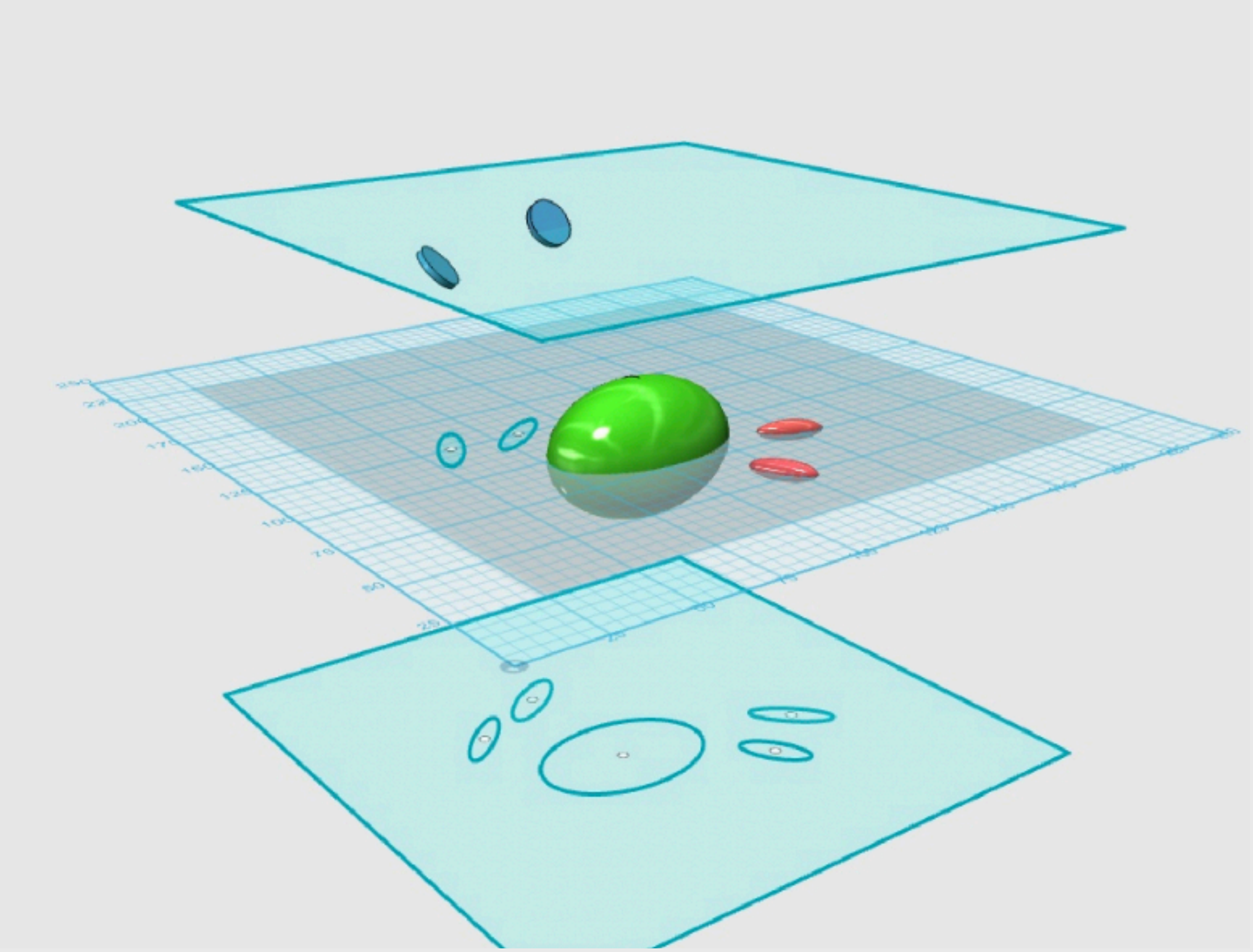
$$\langle \epsilon_i \epsilon_j \rangle = \underbrace{\langle g_i g_j \rangle}_{GG} + \underbrace{\langle \epsilon_i^{(s)} \epsilon_j^{(s)} \rangle}_{II} + \underbrace{\langle \epsilon_i^{(s)} g_j \rangle}_{IG} + \underbrace{\langle g_i \epsilon_j^{(s)} \rangle}_{GI}$$

II term: Correlations between physically close galaxies

- Positive correlation

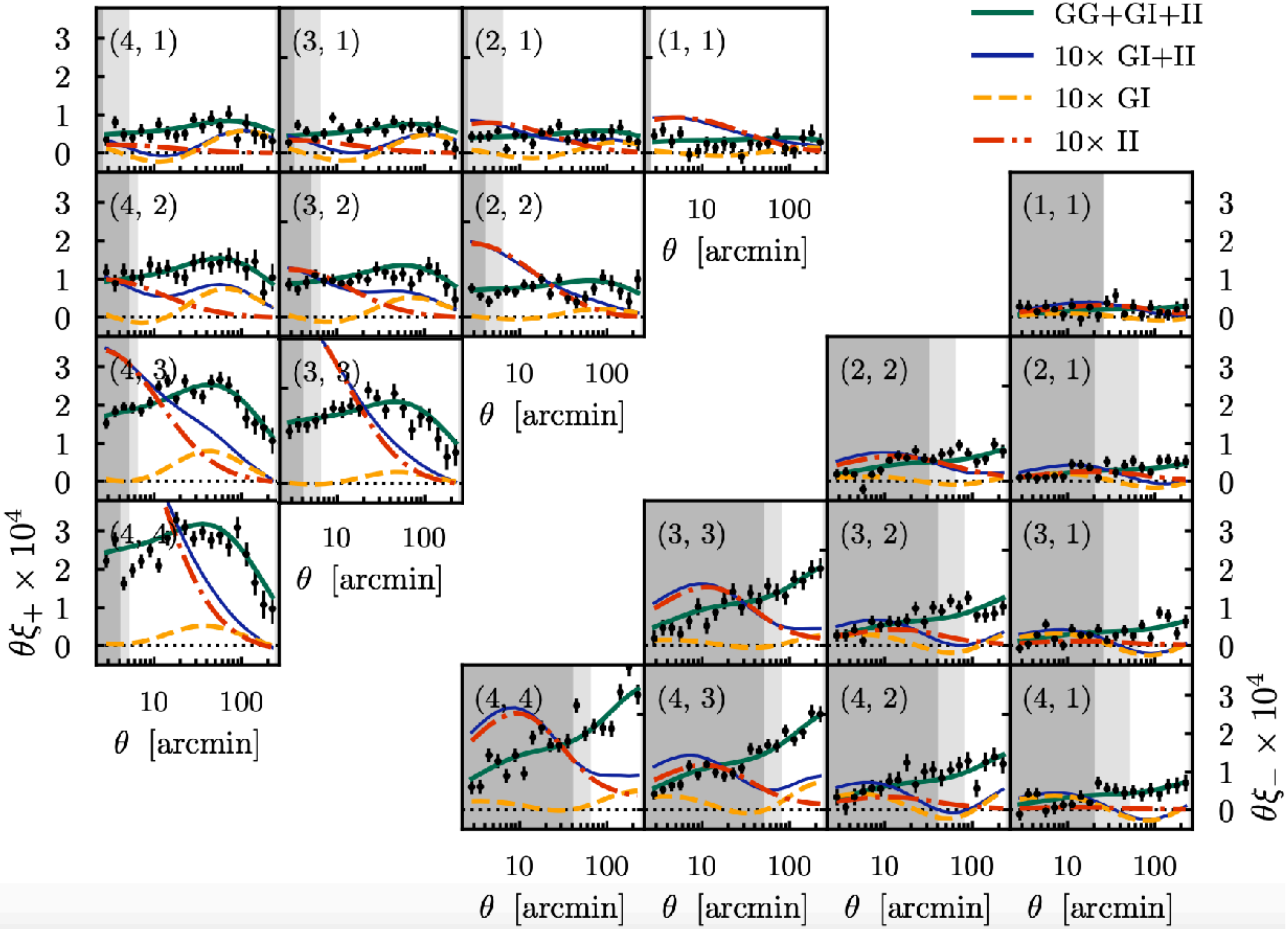
GI term: Correlations between one foreground galaxy and one background galaxy

- Negative correlation



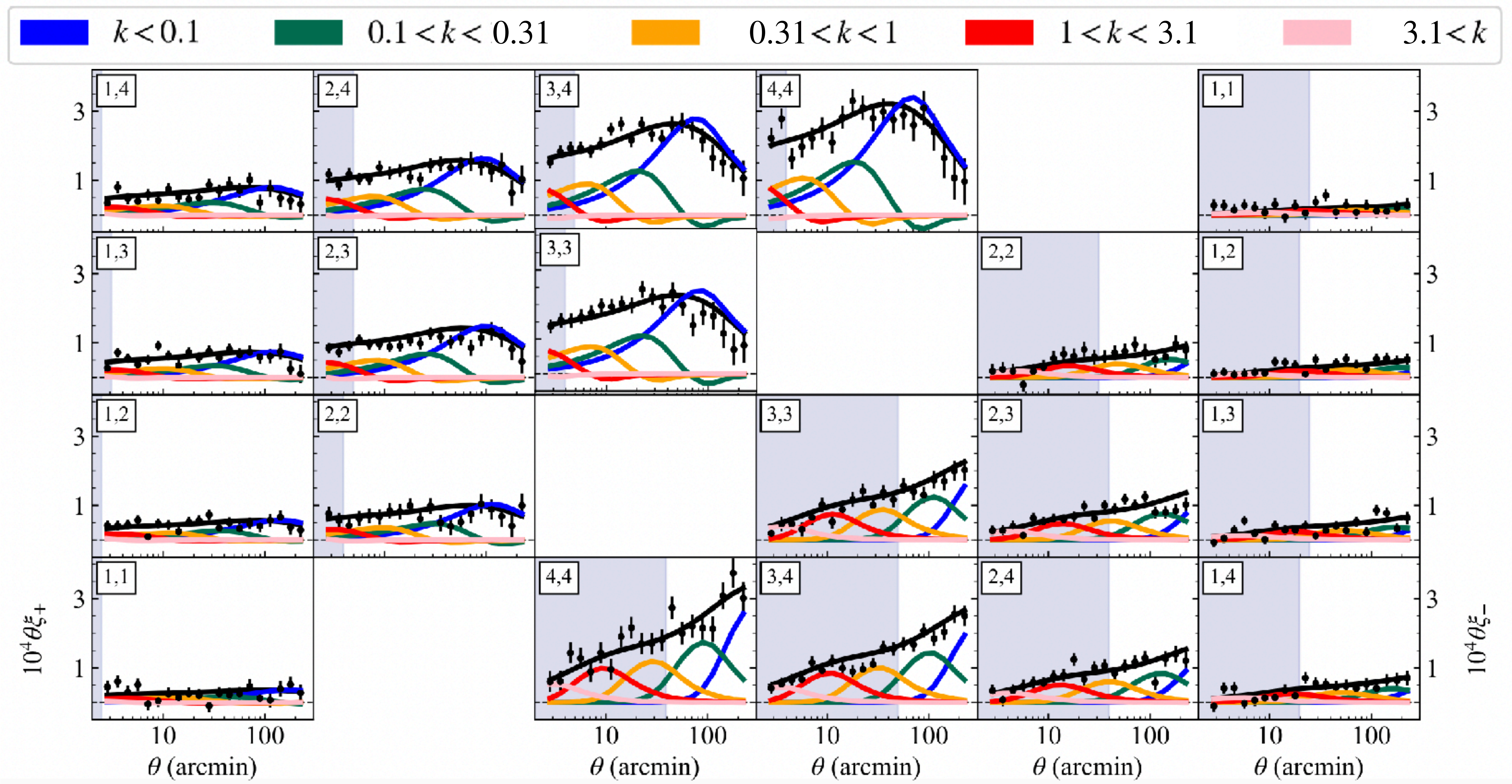
Adapted from
Joachimi et al (2015)

Intrinsic-Alignments



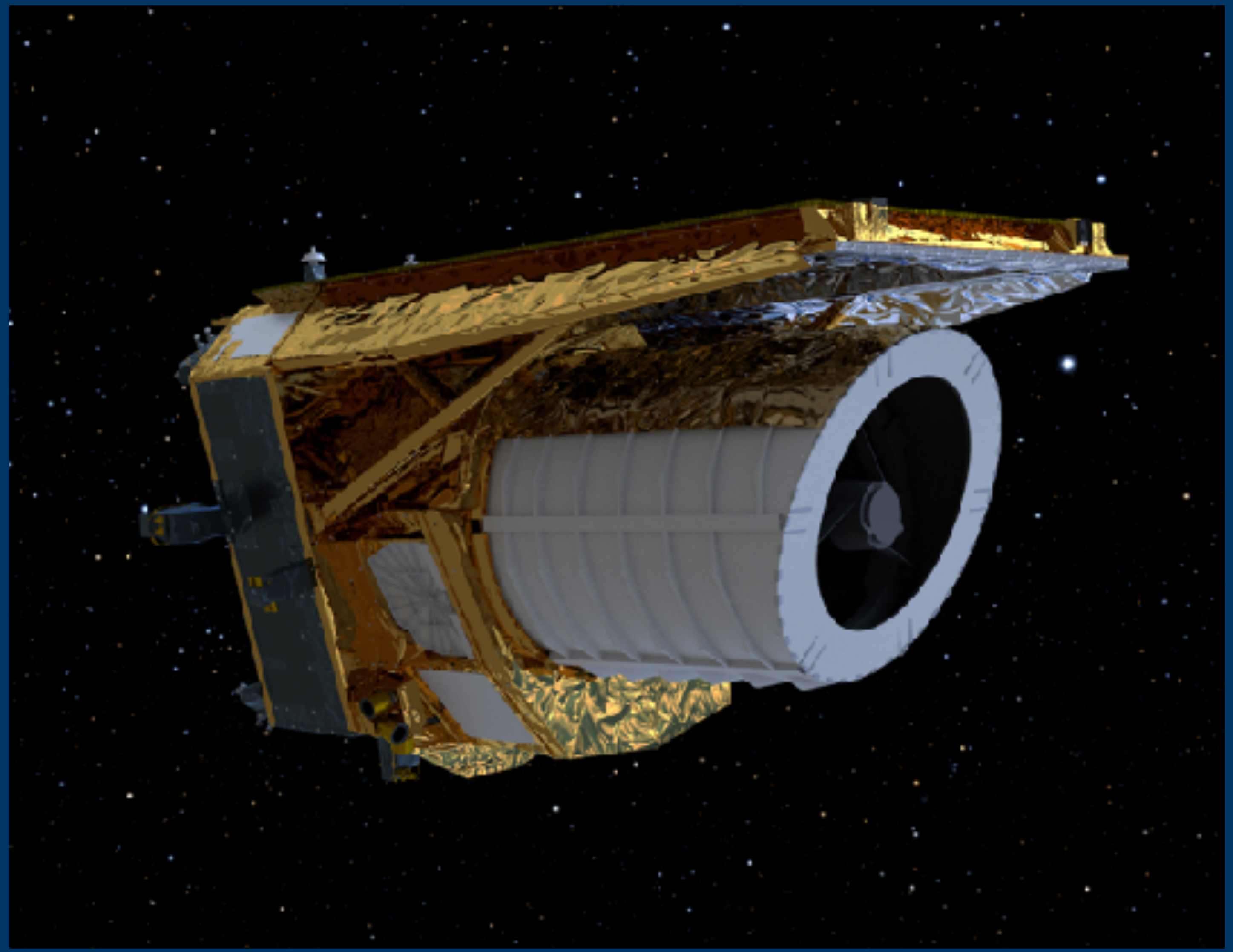
Adapted from
Secco & Samuroff (2021)

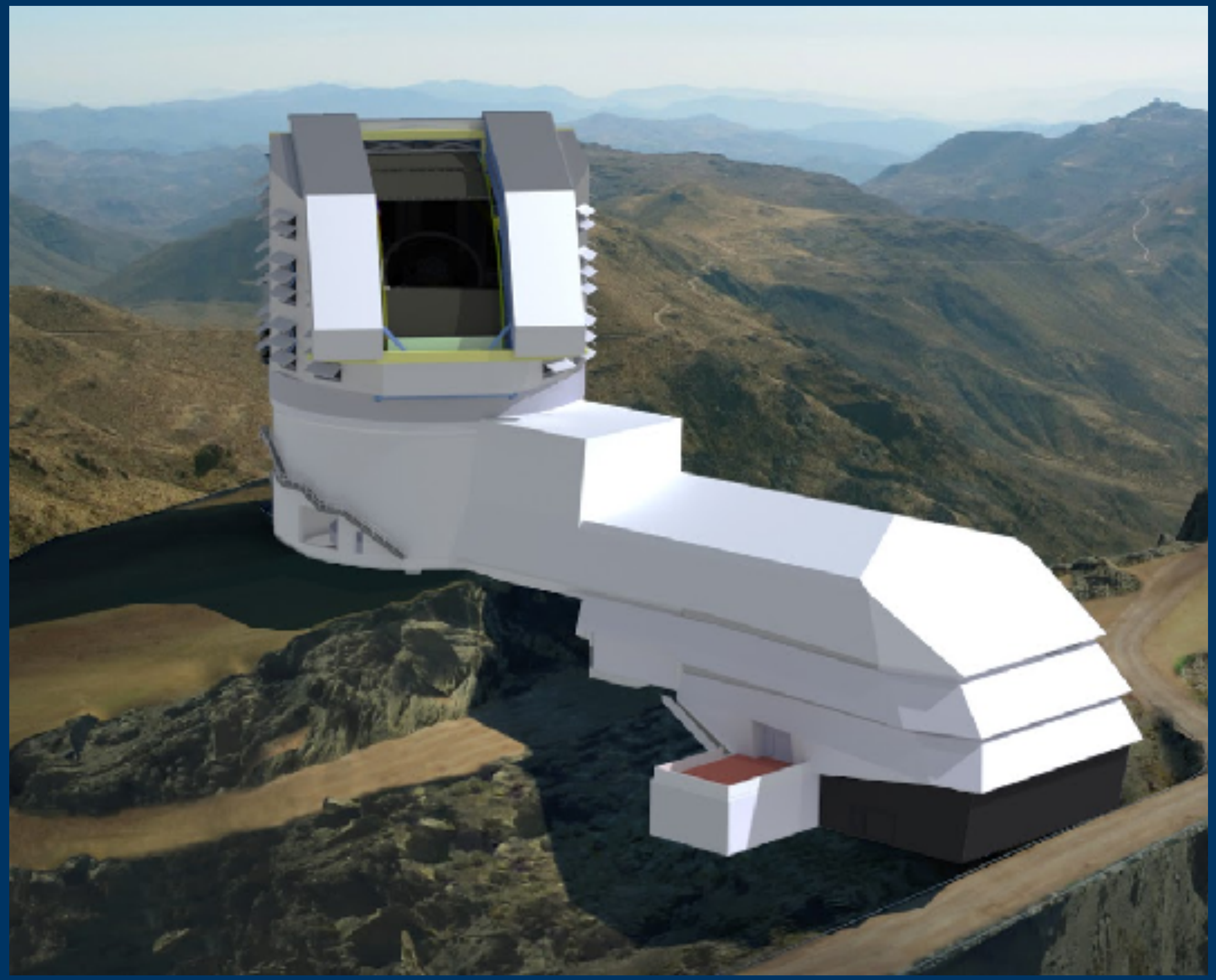
Non-Linearity



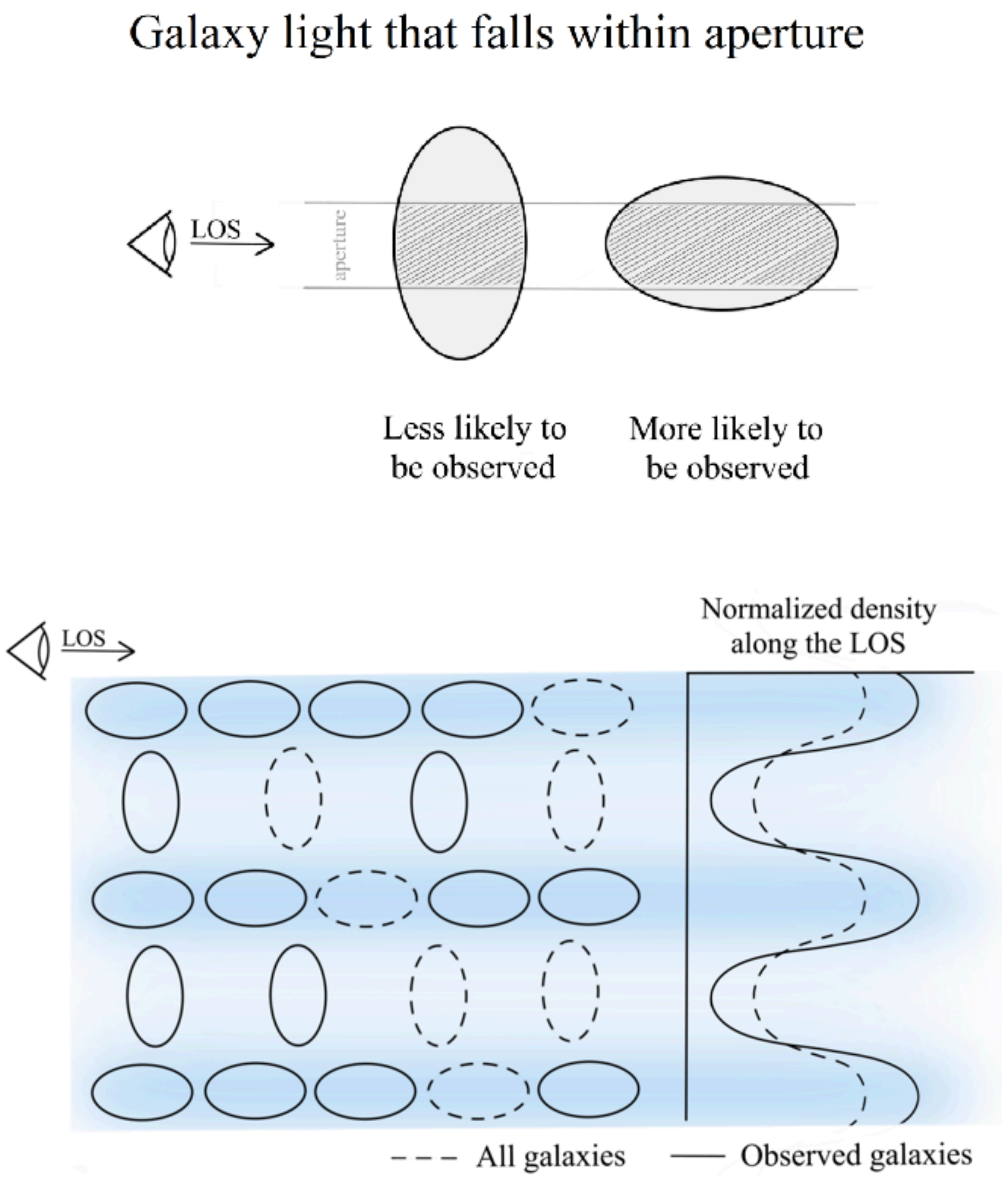
Adapted from
Preston et. al (2023)

Why Should You Care?





Adapted from
Lamman et al (2022)



Alignments probe cosmology

Physics	Proposed	Verified in sims	Constrained from LOWZ
Growth rate	Taruya & Okumura (2020)	X	Okumura & Taruya (2023)
Primordial (anisotropic) non-Gaussianity	Schmidt, Chisari, Dvorkin (2015)	Akitsu+ (2021)	Kurita & Takada (2023)
Primordial magnetic fields	Schmidt, Chisari, Dvorkin (2015) Saga+ (2023)	through PNG only	X
Isotropy	Shiraishi, Okumura, Akitsu (2023)	X	X
BAO	Chisari & Dvorkin (2013)	Okumura, Taruya & Nishimichi (2019)	Xu+ (2023)
Primordial gravitational waves	Schmidt, Pajer, Zaldarriaga (2014) Chisari, Dvorkin, Schmidt (2014)	Akitsu, Li & Okumura (2023)	X
Parity breaking	Biagetti & Orlando (2020)	X	X

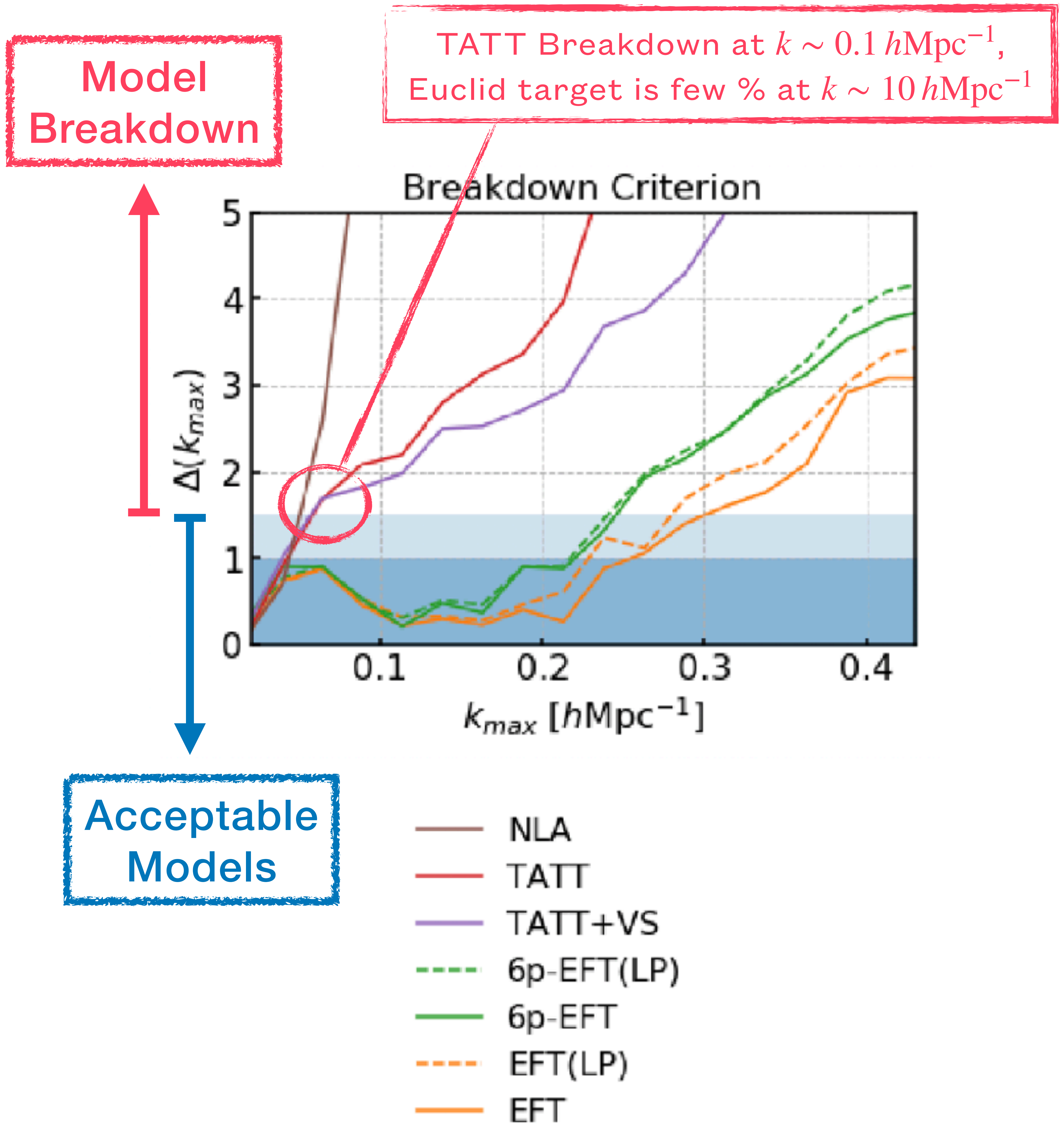
Non-Linearity

To lowest order, the intrinsic shear of the galaxy shapes will be linearly related to the matter tidal field

$$\gamma^I = c_s s_{ij} = c_s \left(\partial_x^2 - \partial_y^2,\, 2\partial_x \partial_y \right) \nabla^{-2} \delta$$

Breaks down quickly at small scales.

EFTofIA can reach $k_{max} = 0.28\, h/\mathrm{Mpc}$ at the expense of adding many free parameters

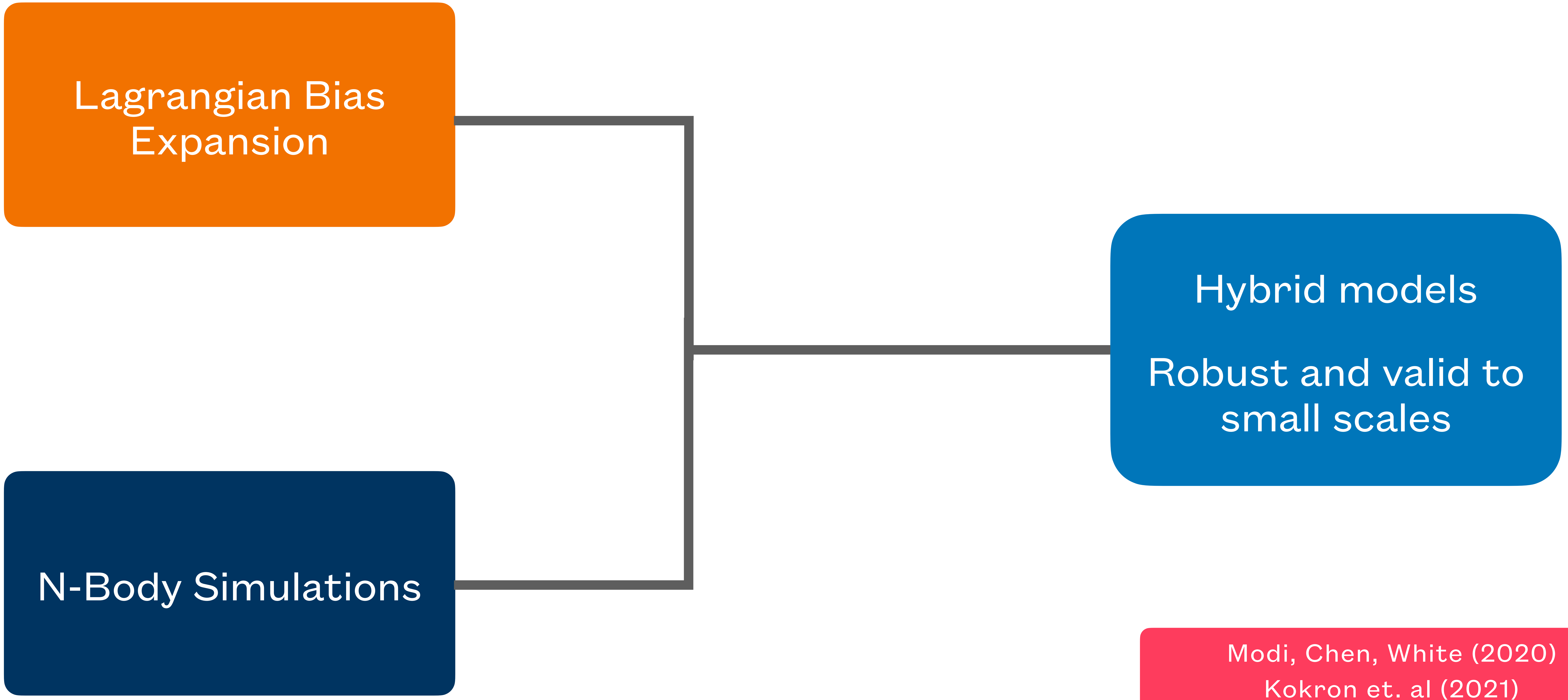


Adapted from Bakx et al (2023)

Simulation-Based Modelling

Hybrid Lagrangian Models

Hybrid Lagrangian Models



Modi, Chen, White (2020)
Kokron et. al (2021)
Zennaro et. al (2021)
Hadzhiyska et al (2021)
Pellejero-Ibáñez,...,**Maion** (2023)
Maion et al (2024)

Bias Expansion

Symmetries and Physical Principles:

- Equivalence Principle (only $\partial^{2n}\Phi$ contributions allowed)
- Statistical Homogeneity
- Statistical Isotropy
- Scalar under rotations

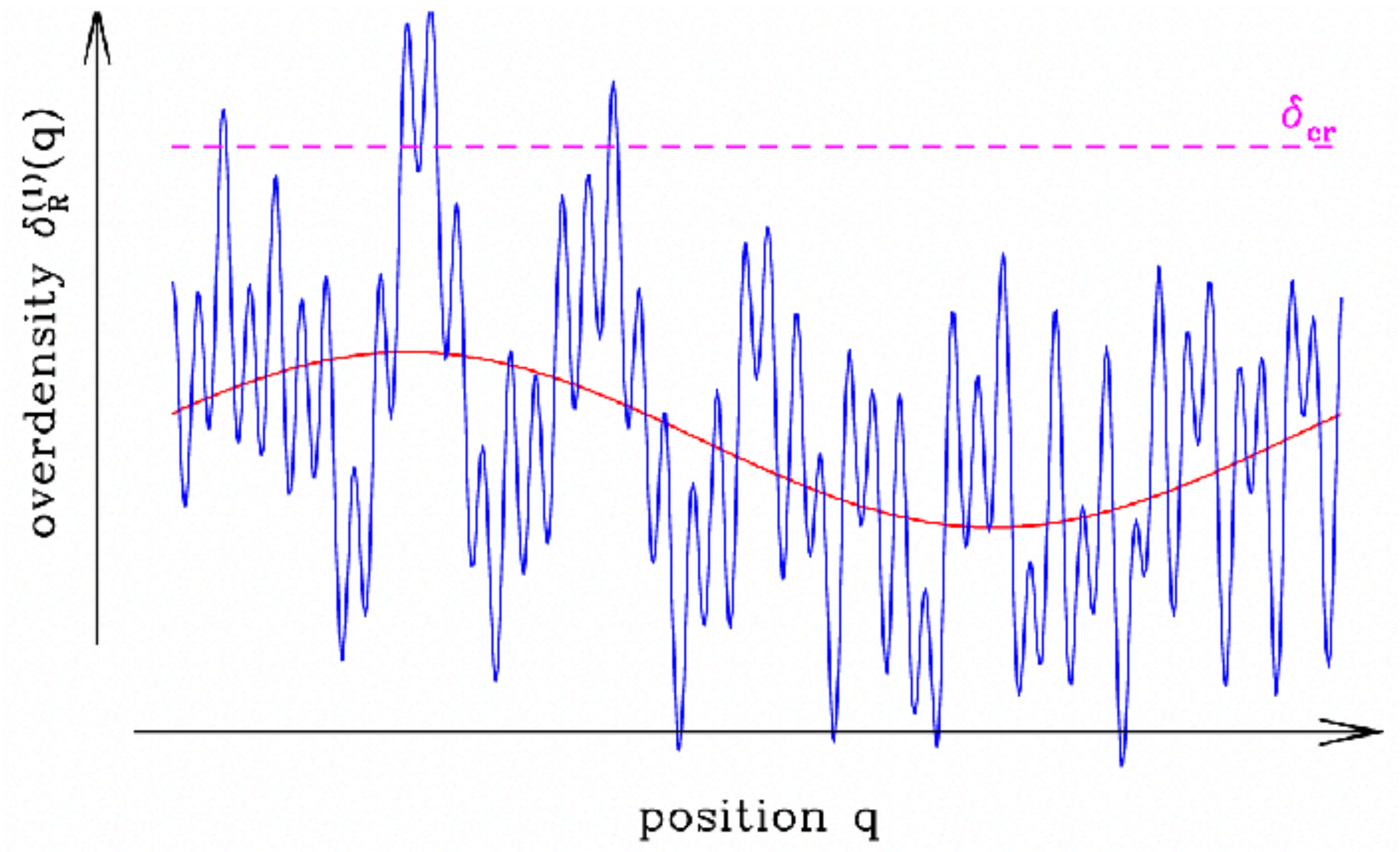
Density:

1^{st} order	: δ
2^{nd} order	: δ^2, s^2
Non-local	: $\nabla^2\delta$
Stochastic	: ε

$$\delta_g = b_1\delta + b_2\delta^2 + b_{s^2}s^2 + b_{\nabla^2}\nabla^2\delta + \varepsilon$$

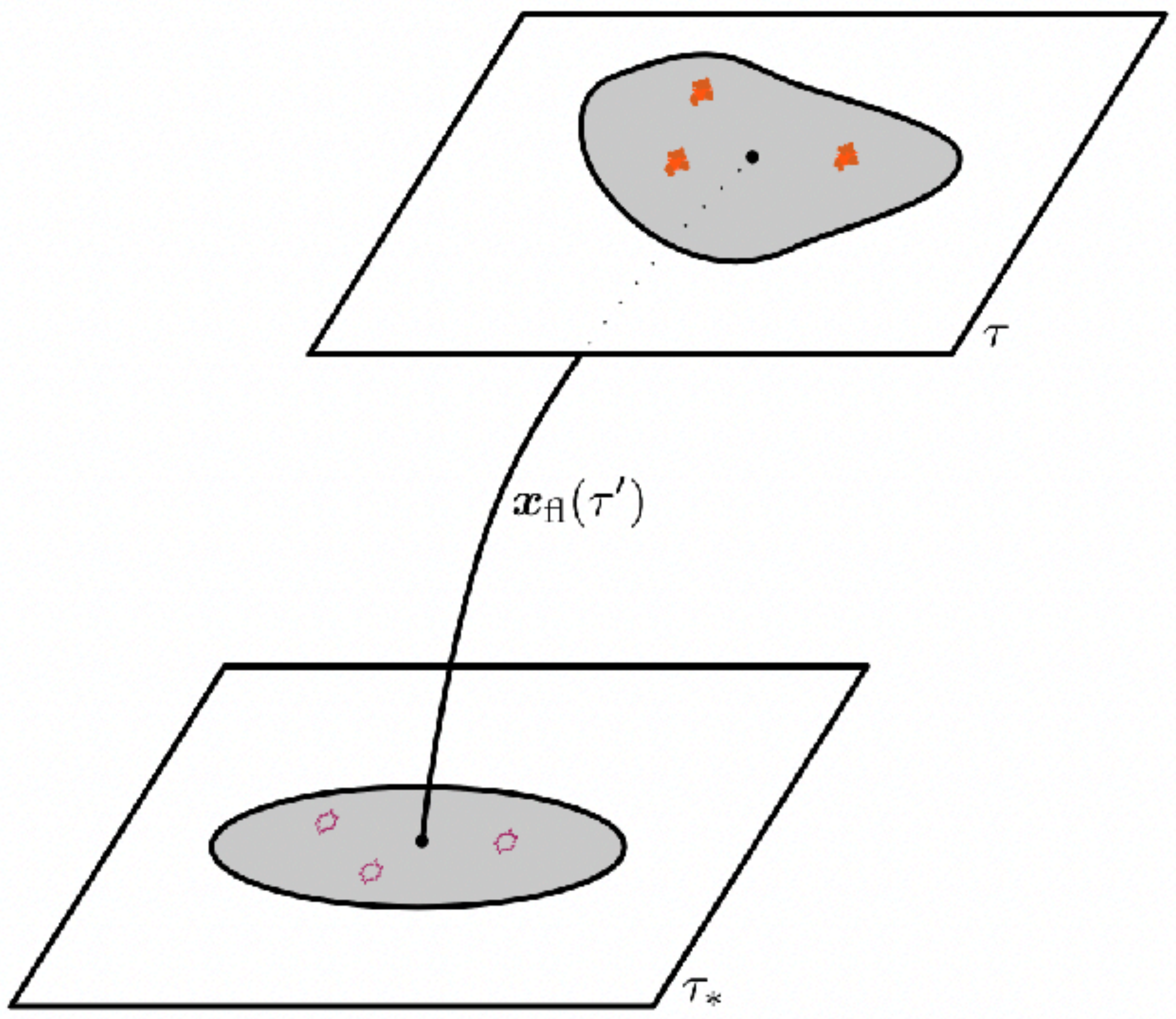
Correlations are setup very early in the universe

Adapted from Desjacques et. al (2016)



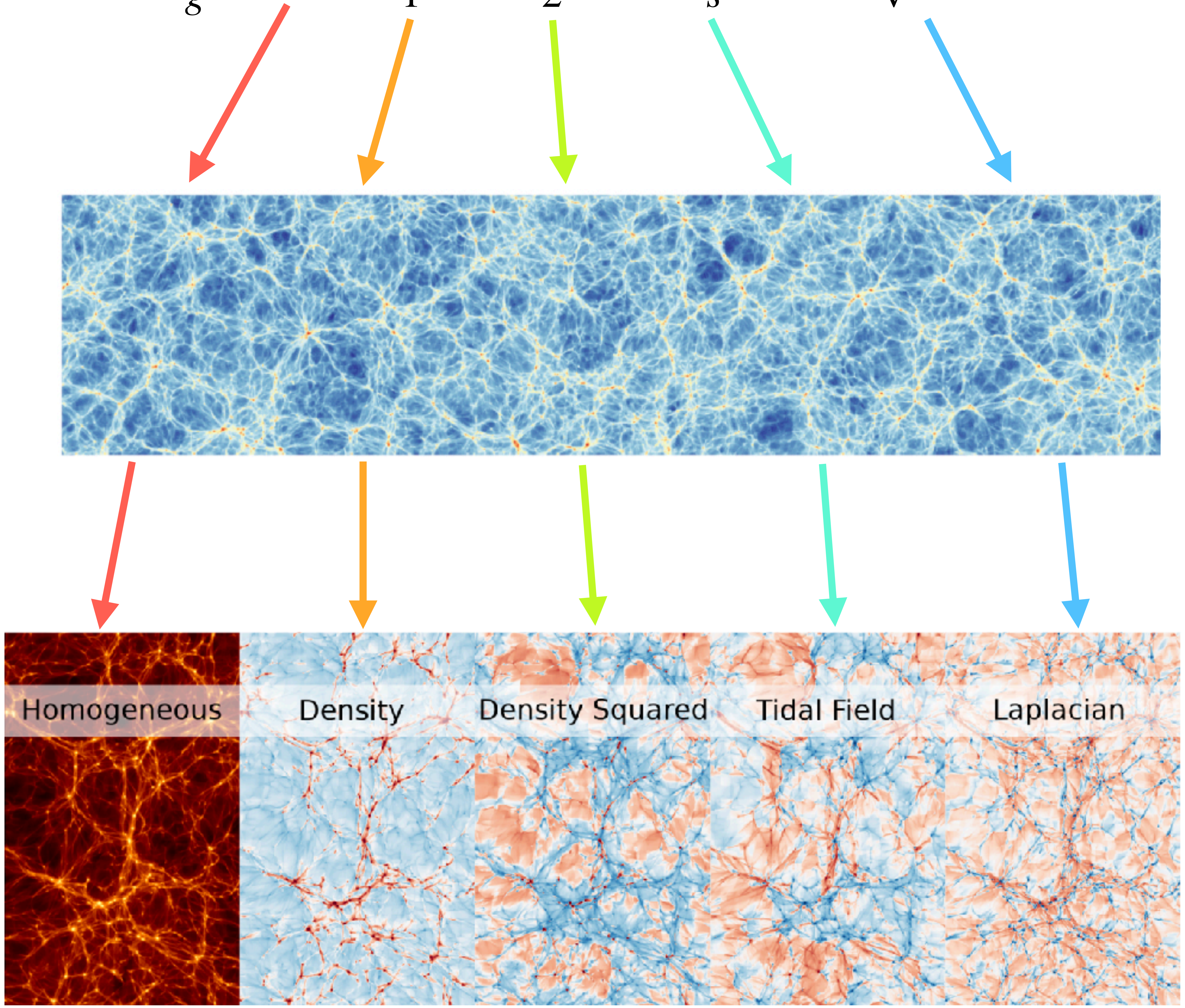
Advection

The modelled galaxy field must be advected from Lagrangian to Eulerian space



Desjacques et. al (2016)

$$1 + \delta_g = 1 + b_1\delta + b_2\delta^2 + b_{s^2}s^2 + b_{\nabla^2}\nabla^2\delta + \varepsilon$$



Zennaro et. al (2021)

Shape Bias-Expansion

Symmetries and Physical Principles:

- Equivalence Principle (only $\partial^{2n}\Phi$ contributions allowed)
- Statistical Homogeneity
- Statistical Isotropy
- Rank-2 tensor under rotations

$$g_{ij} = c_s s_{ij} + c_{s\delta}\delta s_{ij} + c_{s\otimes s}(s\otimes s)_{ij} + c_t t_{ij} + c_{\nabla^2}\nabla^2 s_{ij} + \varepsilon_{ij}$$

Shapes:

- 1st order : s_{ij}
- 2nd order : $(s\otimes s)_{ij}, \delta s_{ij}, t_{ij}$
- Non-local : $\nabla^2 s_{ij}$
- Stochastic : ε_{ij}

$$(s\otimes s)_{ij} = \left(s_{il}s_{lj} - \delta^K_{ij}\frac{s^2}{3} \right)$$

$$t_{ij} = \left(\frac{\partial_i\partial_j}{\nabla^2} - \frac{1}{3}\delta^K_{ij} \right)\left(\theta(\mathbf{x}) - \delta(\mathbf{x})\right)$$

HYMALAIA

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HYMALAIA: a hybrid lagrangian model for intrinsic alignments

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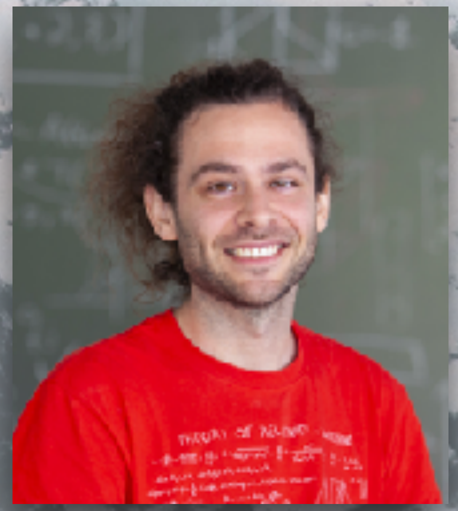
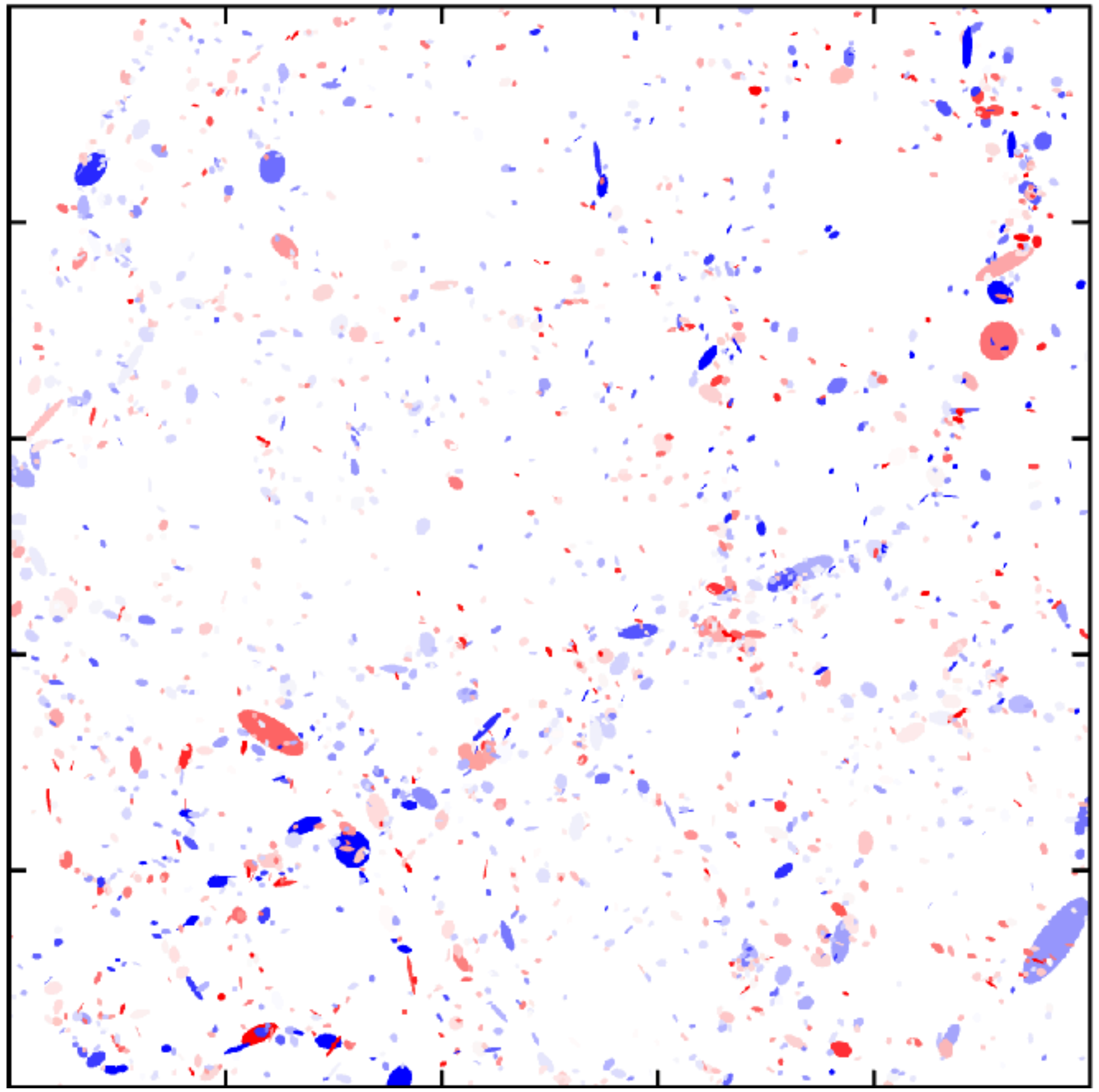
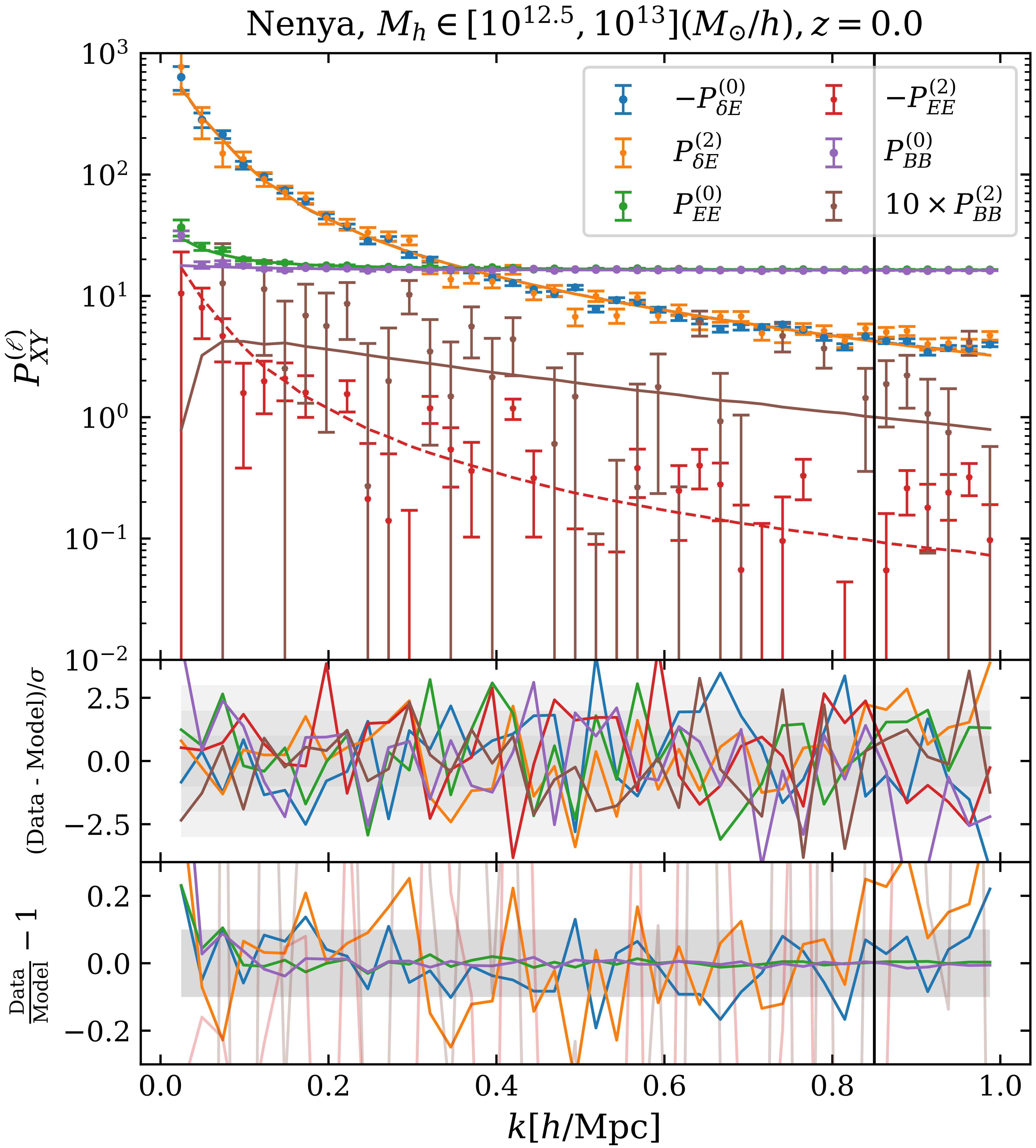
Accepted 2024 May 21. Received 2024 May 21; in original form 2023 August 1



HYMALAIA




Nenya, $M_h \in [10^{12.5}, 10^{13}](M_\odot/h), z = 0.0$
$-P^{(0)}_{\delta E}$
$P^{(2)}_{\delta E}$
$P^{(0)}_{EE}$
$-P^{(2)}_{EE}$
$P^{(0)}_{BB}$
$10 \times P^{(2)}_{BB}$
$P^{(\ell)}_{XY}$
(Data - Model)/σ
$\frac{\text{Data}}{\text{Model}} - 1$
$k[h/\text{Mpc}]$
10^{3} 10^{2} 10^{1} 10^{0} 10^{-1} 10^{-2}
2.5 0.0 −2.5
0.2 0.0 −0.2
0.0 0.2 0.4 0.6 0.8 1.0

Model Validation

To evaluate the performance of the model we will use the reduced chi-squared,

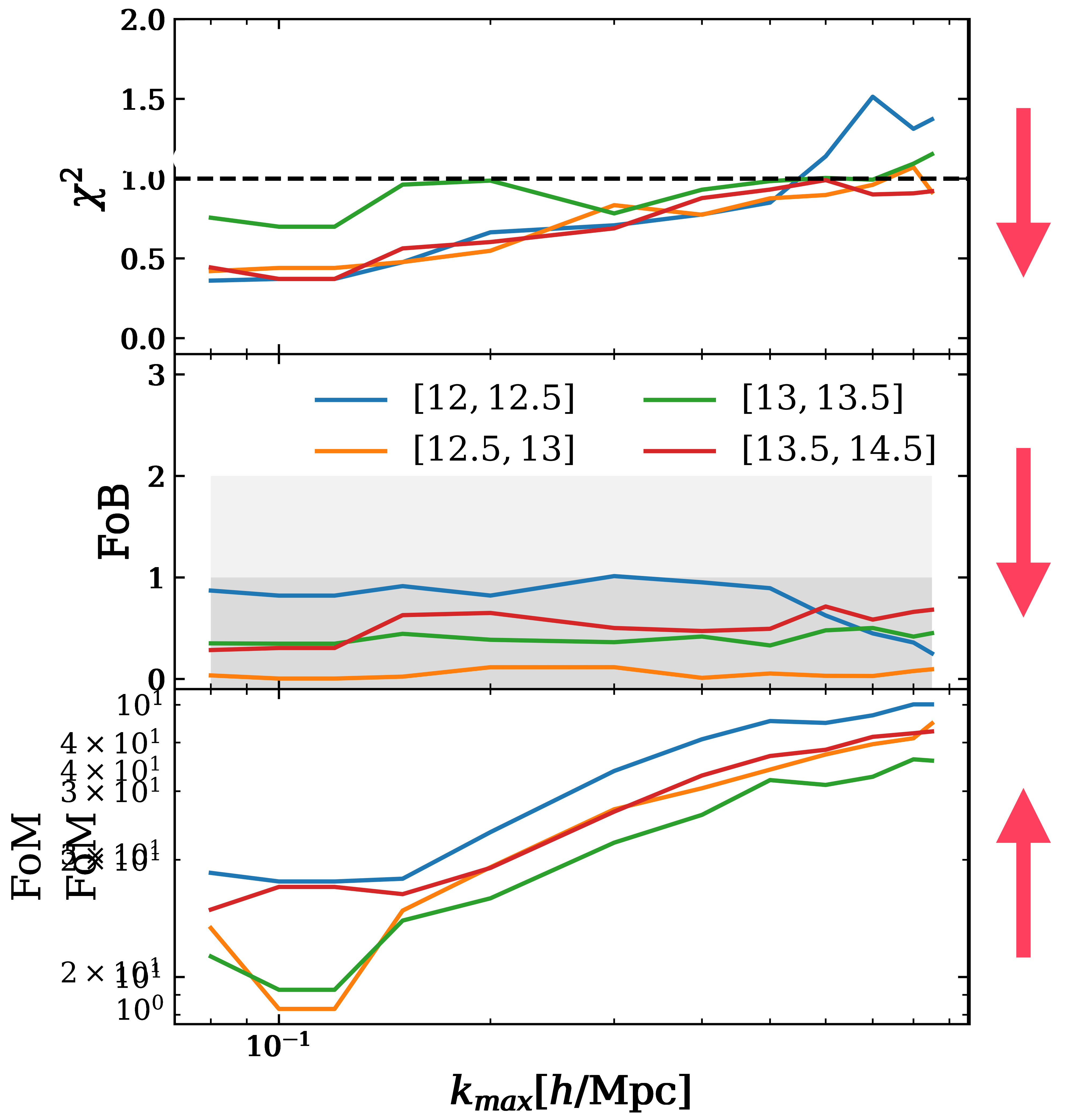
$$\chi^2_{\rm red} = \frac{1}{N_{\rm dof}} \sum_{\ell,\ell'=0,2} \sum_{\alpha,\beta} \sum_{i,j} \left(P^{(\ell)}_{\alpha}(k_i,\Theta) - \widehat{P}^{(\ell)}_{\alpha}(k_i)\right)\left[C^{\ell,\ell'}_{\alpha,\beta}\right]^{-1}_{ij}\left(P^{(\ell')}_{\beta}(k_j,\Theta) - \widehat{P}^{(\ell')}_{\beta}(k_j)\right)$$

the Figure of Bias, defined as

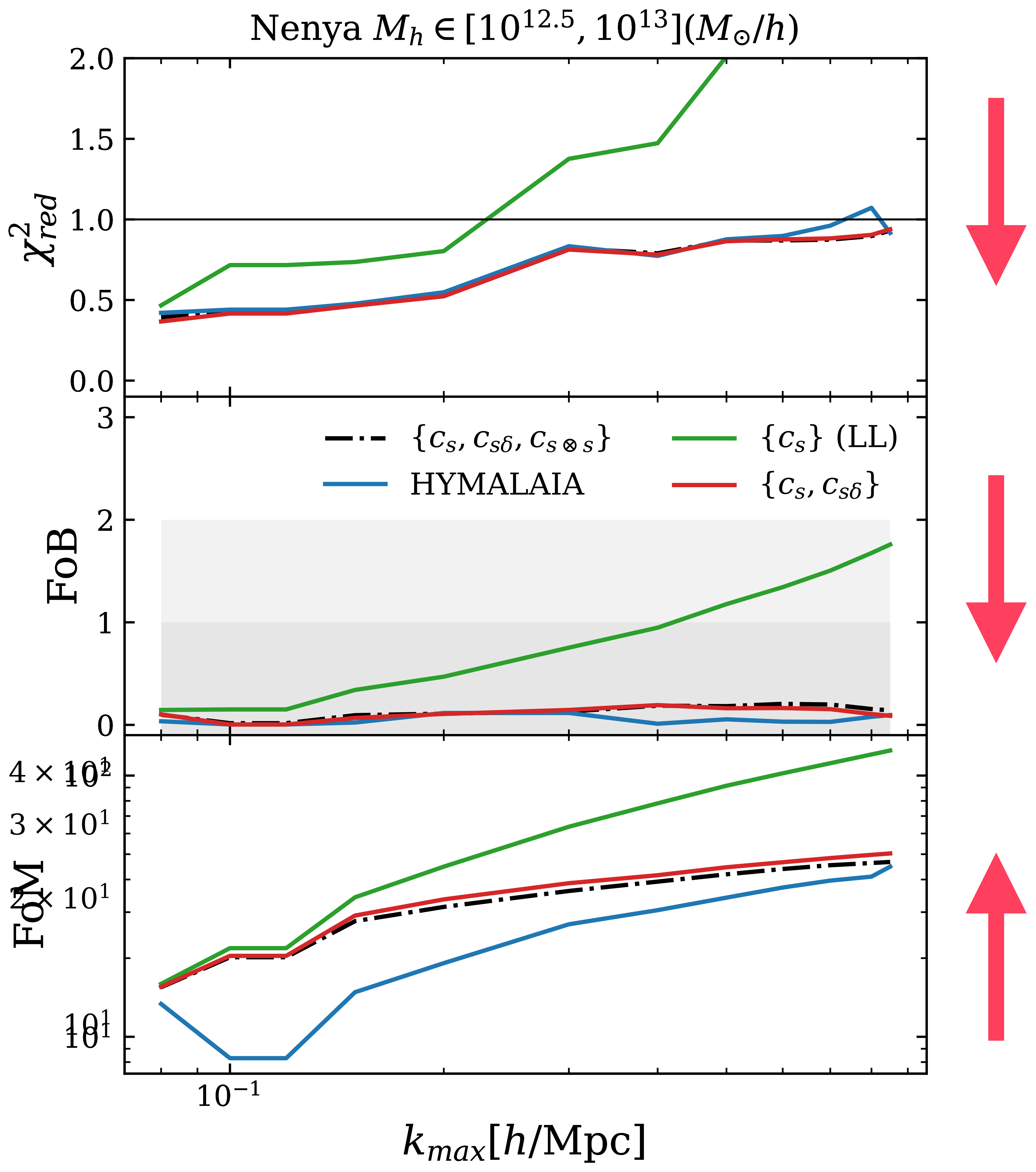
$$\mathrm{FoB}(\mathrm{k}_{\rm max}) = \frac{\left|\mathrm{c}_{\rm s}^{\rm fid} - \mathrm{c}_{\rm s}(\mathrm{k}_{\rm max})\right|}{\sqrt{\sigma^2_{\rm fid} + \sigma^2_{\mathrm{c_s}}(\mathrm{k}_{\rm max})}}$$

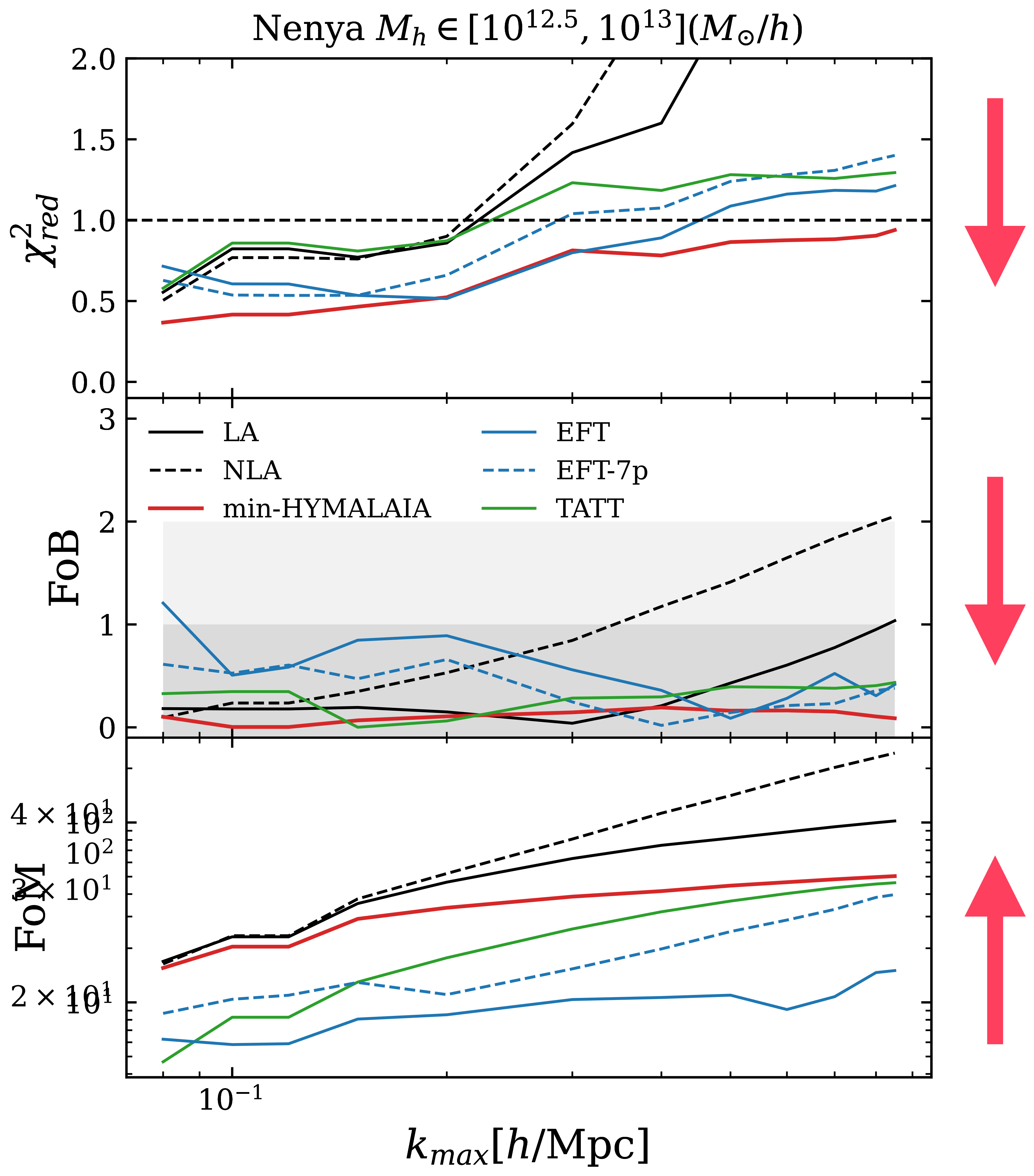
and the Figure of Merit, given by

$$\mathrm{FoM} = \sqrt{\det\left[\frac{\Theta_{\alpha\beta}}{\theta^{\rm fid}_{\alpha}\theta^{\rm fid}_{\beta}}\right]^{-1}}$$



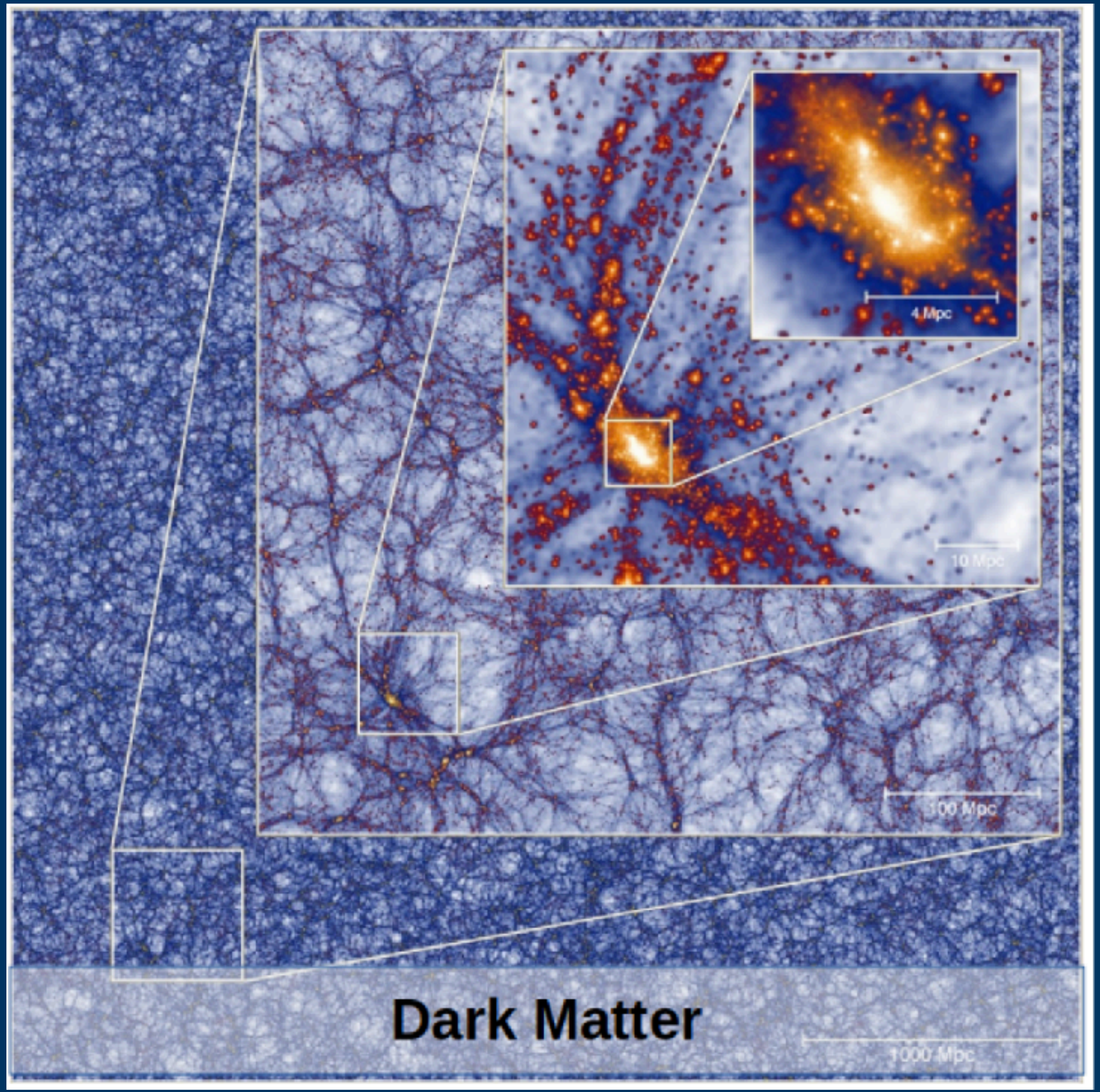
Model Validation



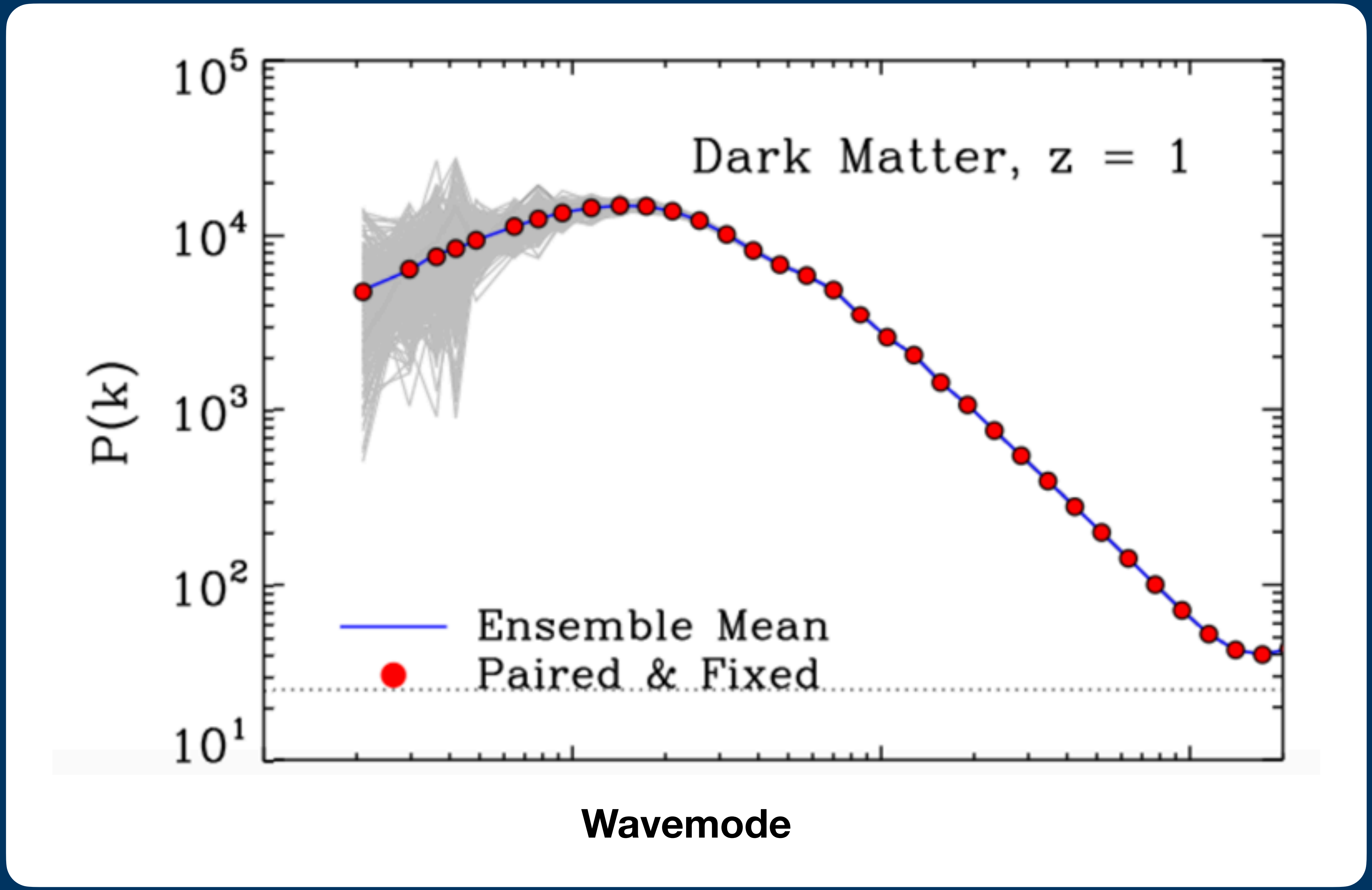


Variance Reduction

Variance Reduction



MXXL Simulation (Angulo et al 2013)



Angulo & Pontoon (2016)

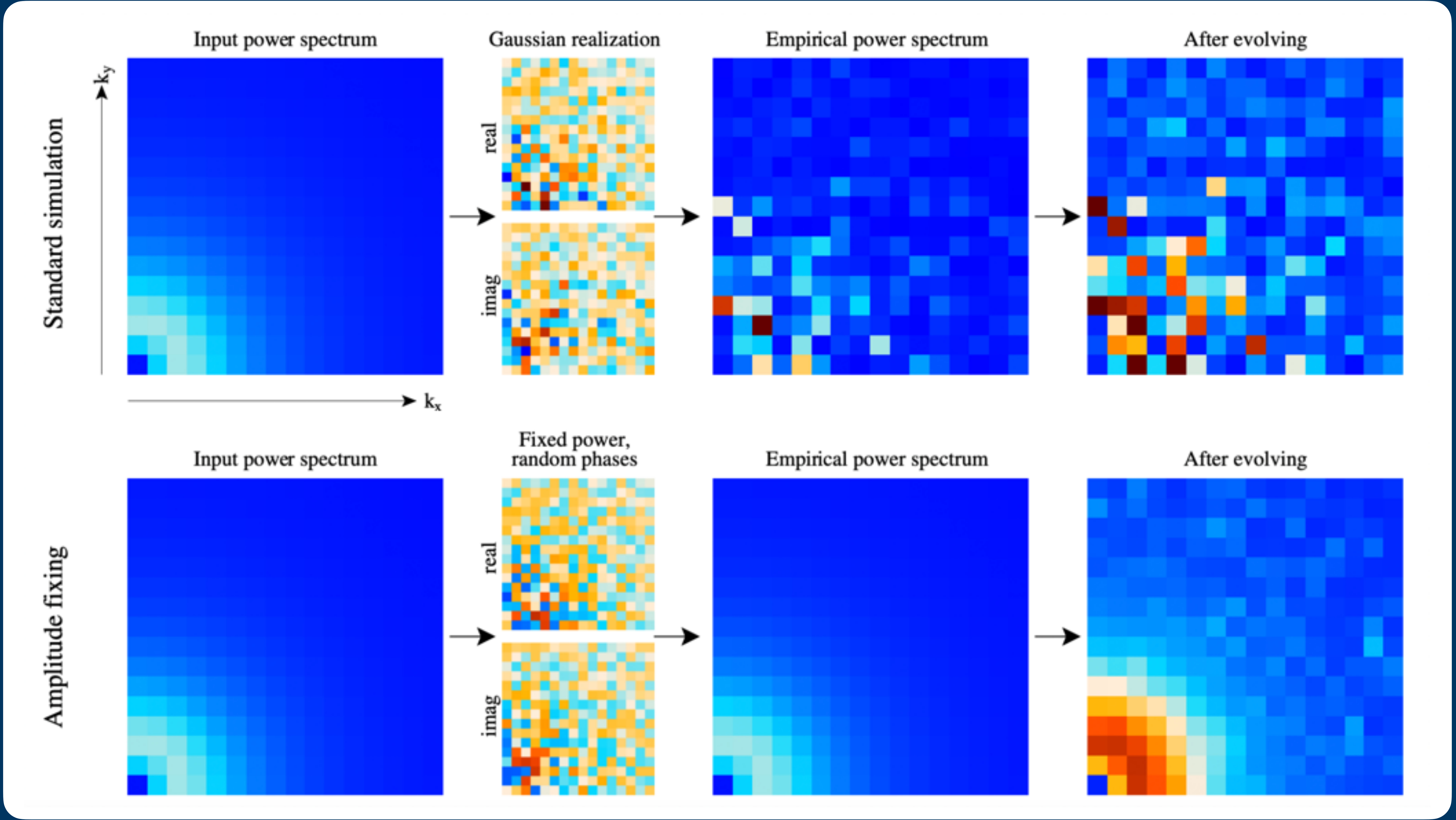
Fixing

$$\mathscr{P}(|\delta(\mathbf{k})|, \theta_{\mathbf{k}}) = \frac{|\delta|}{L^3 P} e^{-|\delta|^2/L^3 P}$$

Fix amplitudes of the initial modes to:

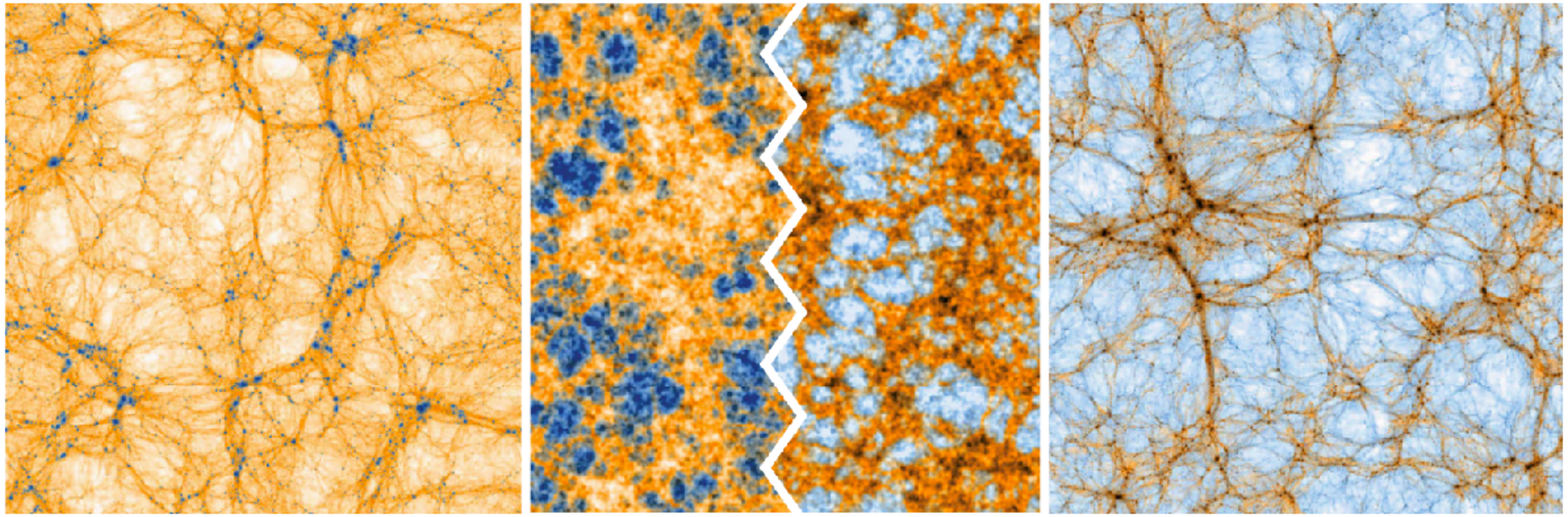
$$|\delta_L(\mathbf{k})| = \sqrt{P(k)} \qquad \theta(\mathbf{k}) \in [0, 2\pi]$$

$$\delta(\mathbf{k})\delta(-\mathbf{k}) = \sqrt{P(k)}e^{i\theta(\mathbf{k})}\sqrt{P(k)}e^{-i\theta(\mathbf{k})} = P(k)$$



Villaescusa-Navarro (2018)

Pairing



Pontzen et al (2016)

$$\delta_A(\mathbf{k}) = \sqrt{P(k)}e^{i\theta(\mathbf{k})}$$

$$\delta_B(\mathbf{k}) = \sqrt{P(k)}e^{i(\theta(\mathbf{k})+\pi)} = -\delta_A(\mathbf{k})$$

F&P

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Statistics of biased tracers in variance-suppressed simulations

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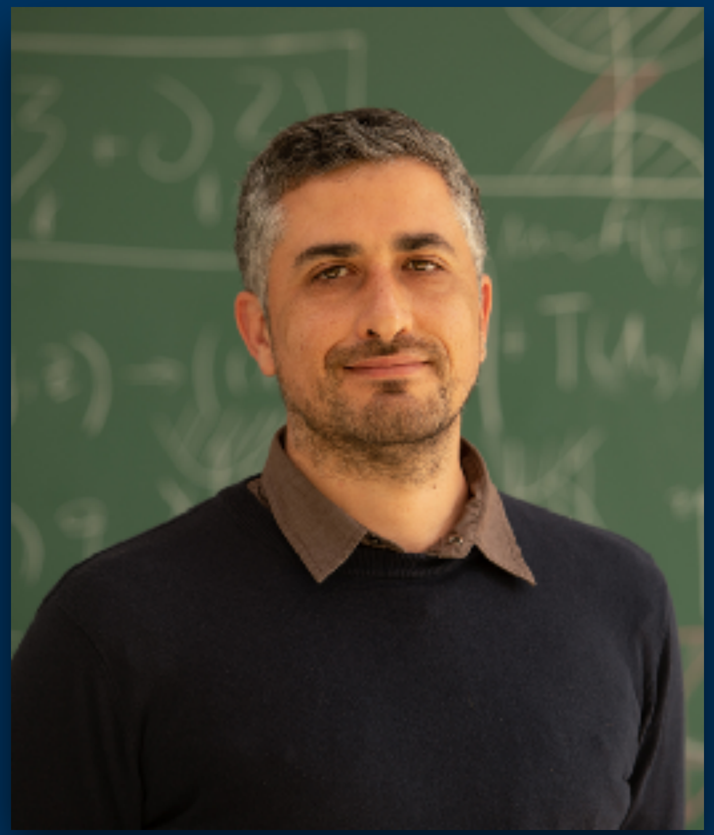
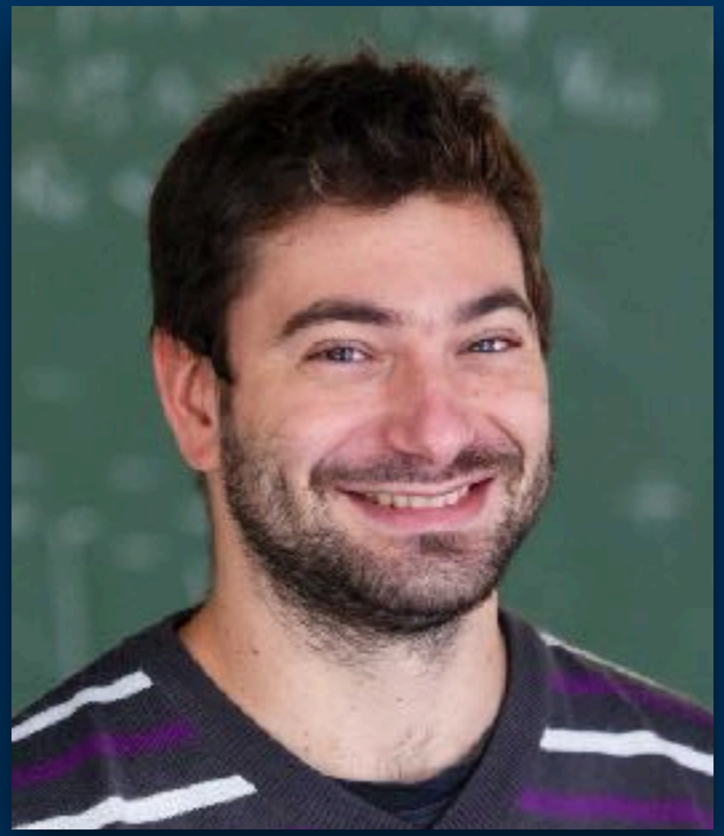
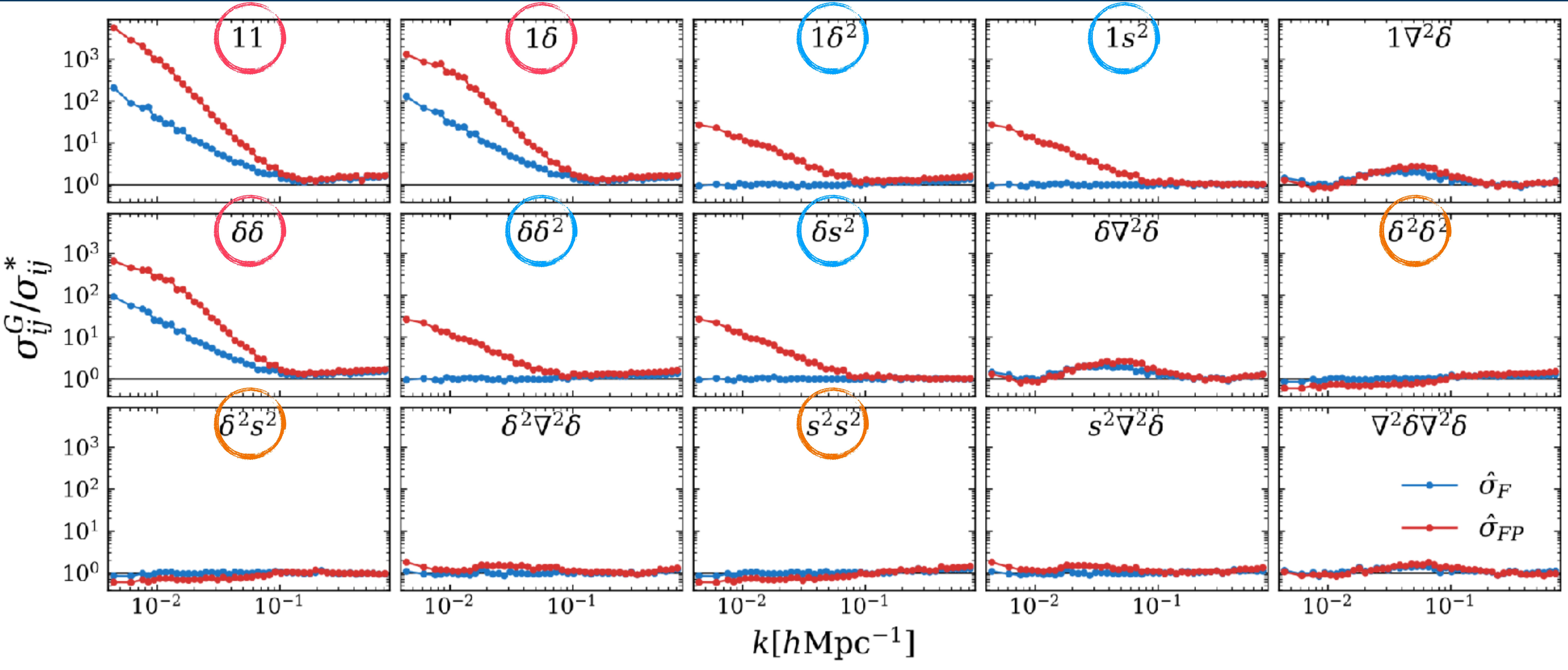


Raul Angulo



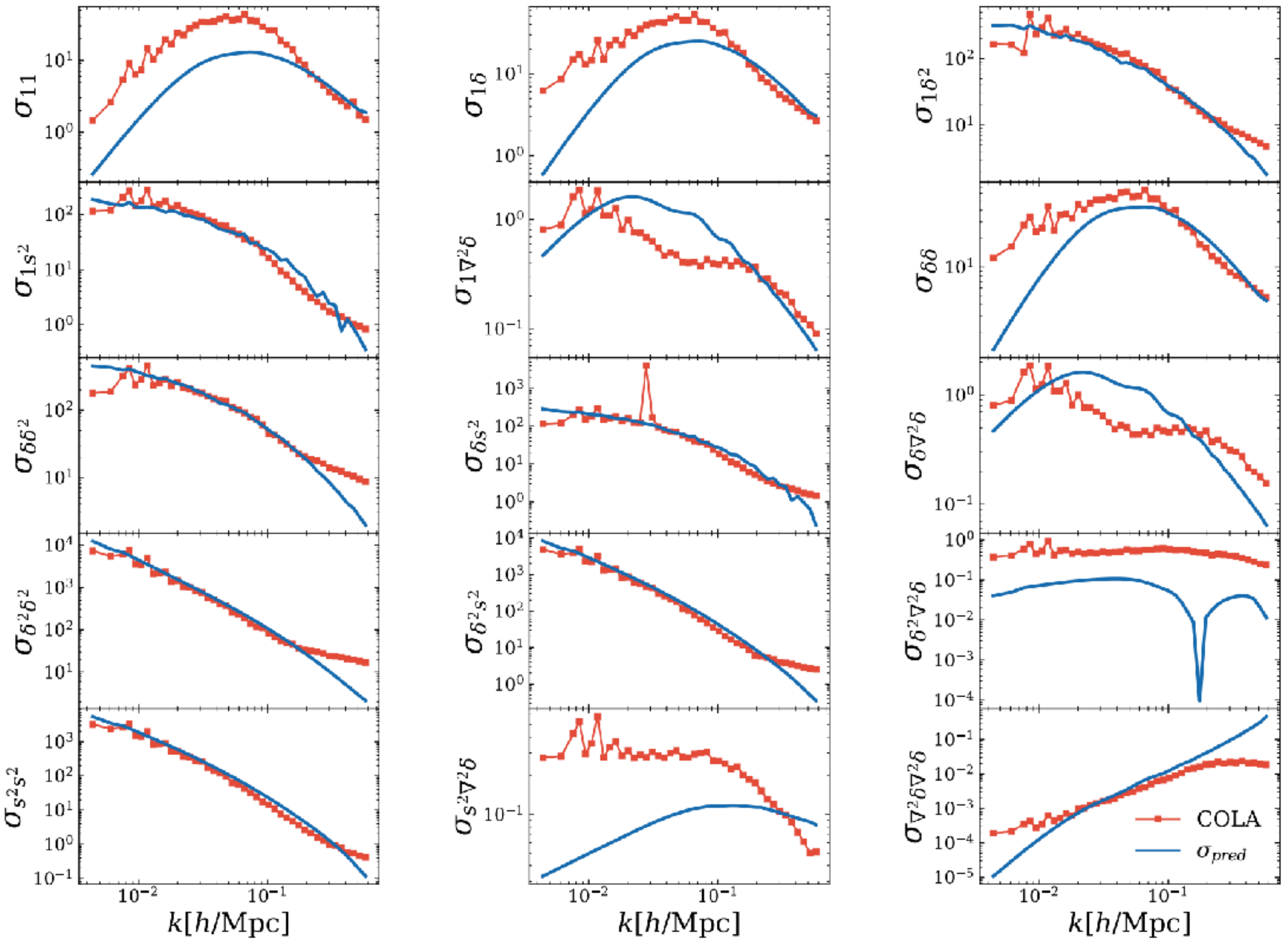
Matteo Zennaro

COLA Simulations

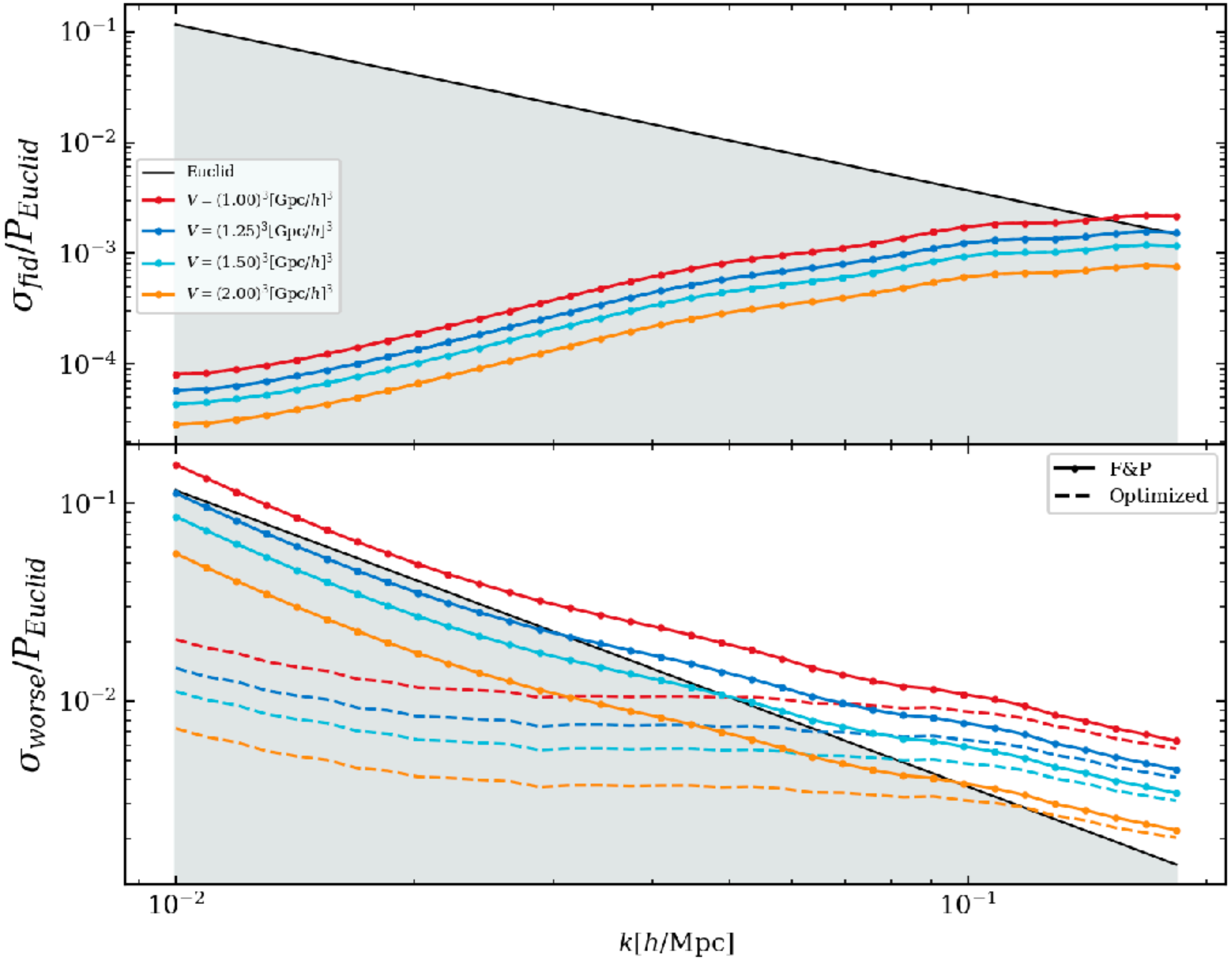


- Both Methods
- Just Pairing
- None

Variance Predictions

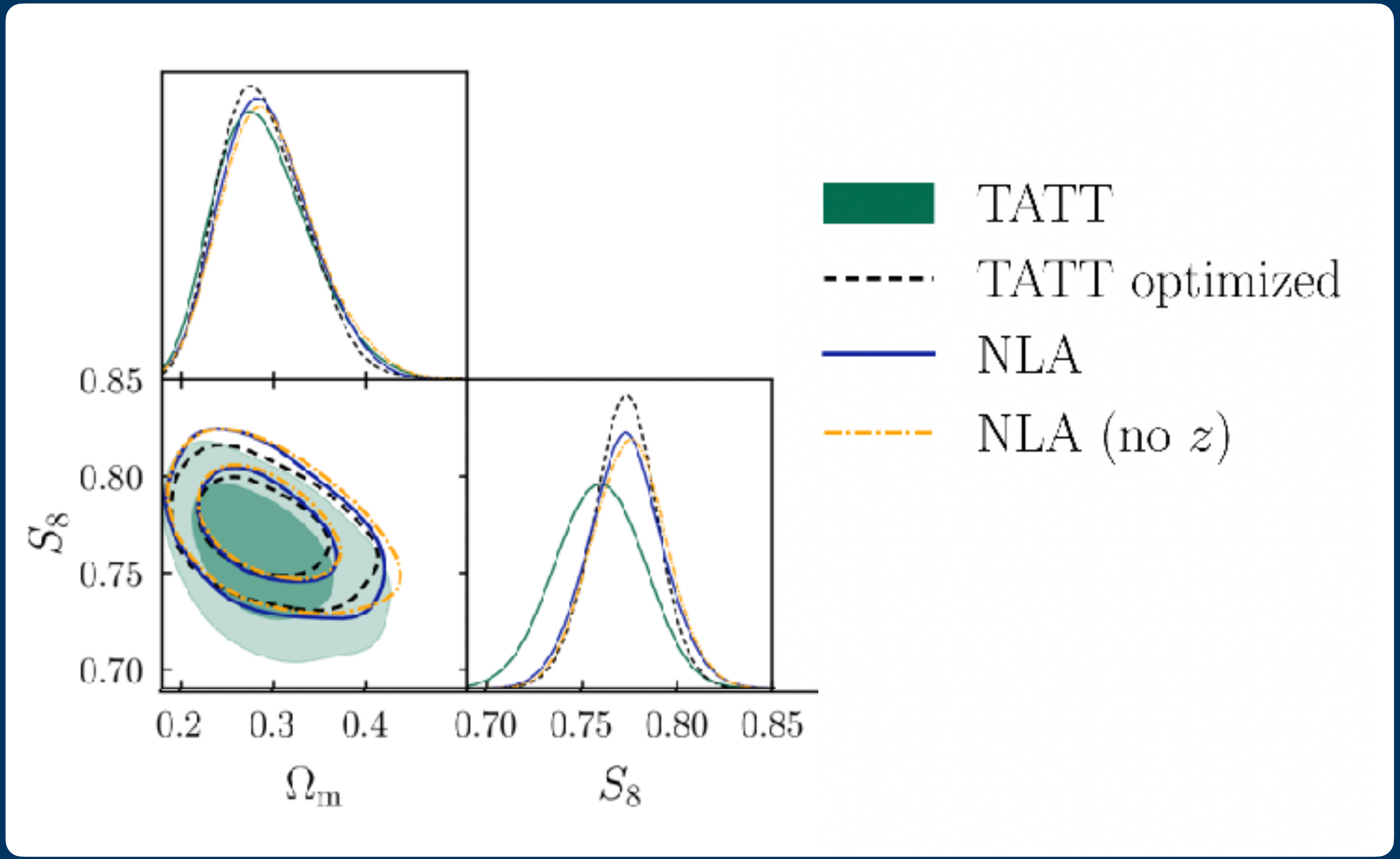


Model Precision

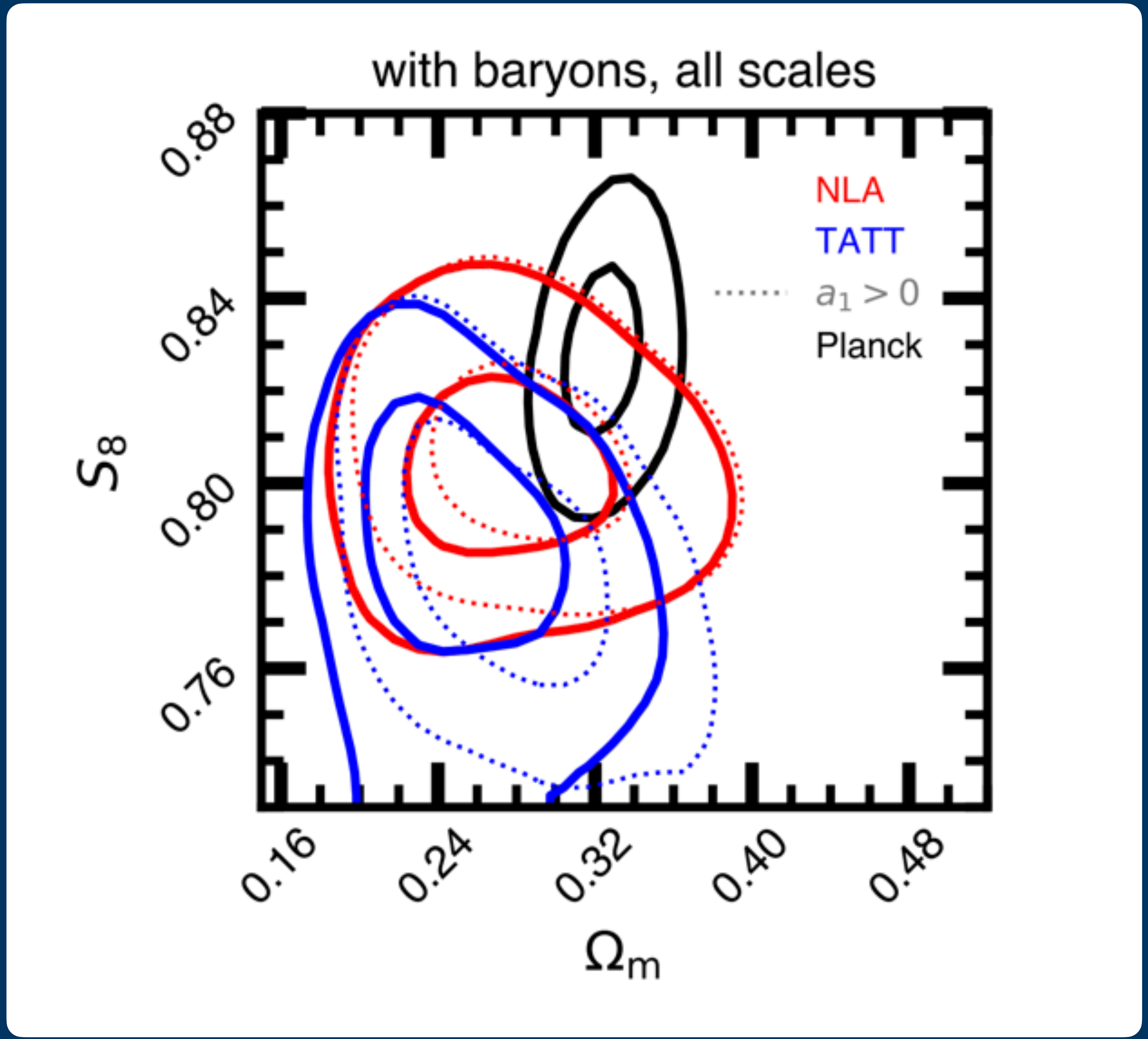


A simulation with mere 20% of the volume of one Euclid survey redshift slice is sufficient

Priors on Bias



Secco & Samuroff (2021)



Aricò et al (2021)

Bias Measurements

Probabilistic Shape Bias

Astronomy & Astrophysics manuscript no. output
September 23, 2024

Probabilistic Estimators of Lagrangian Shape Biases: Universal Relations and Physical Insights

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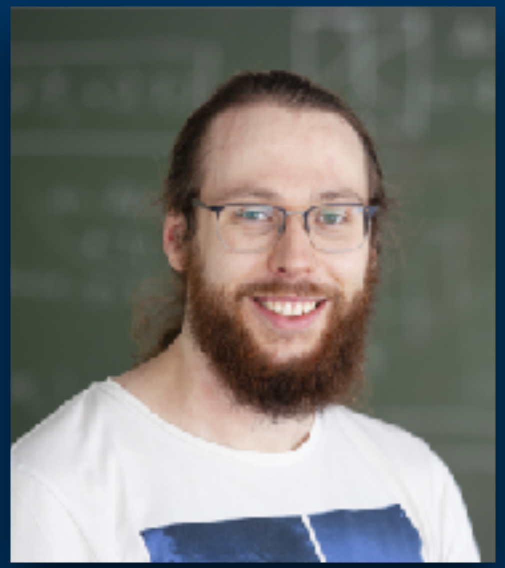
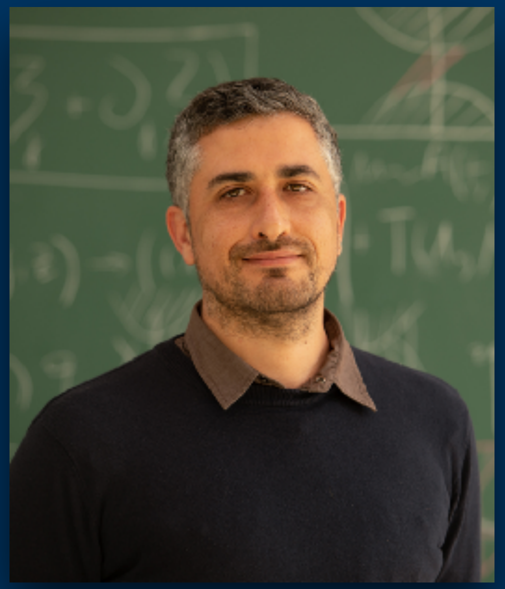
September 23, 2024

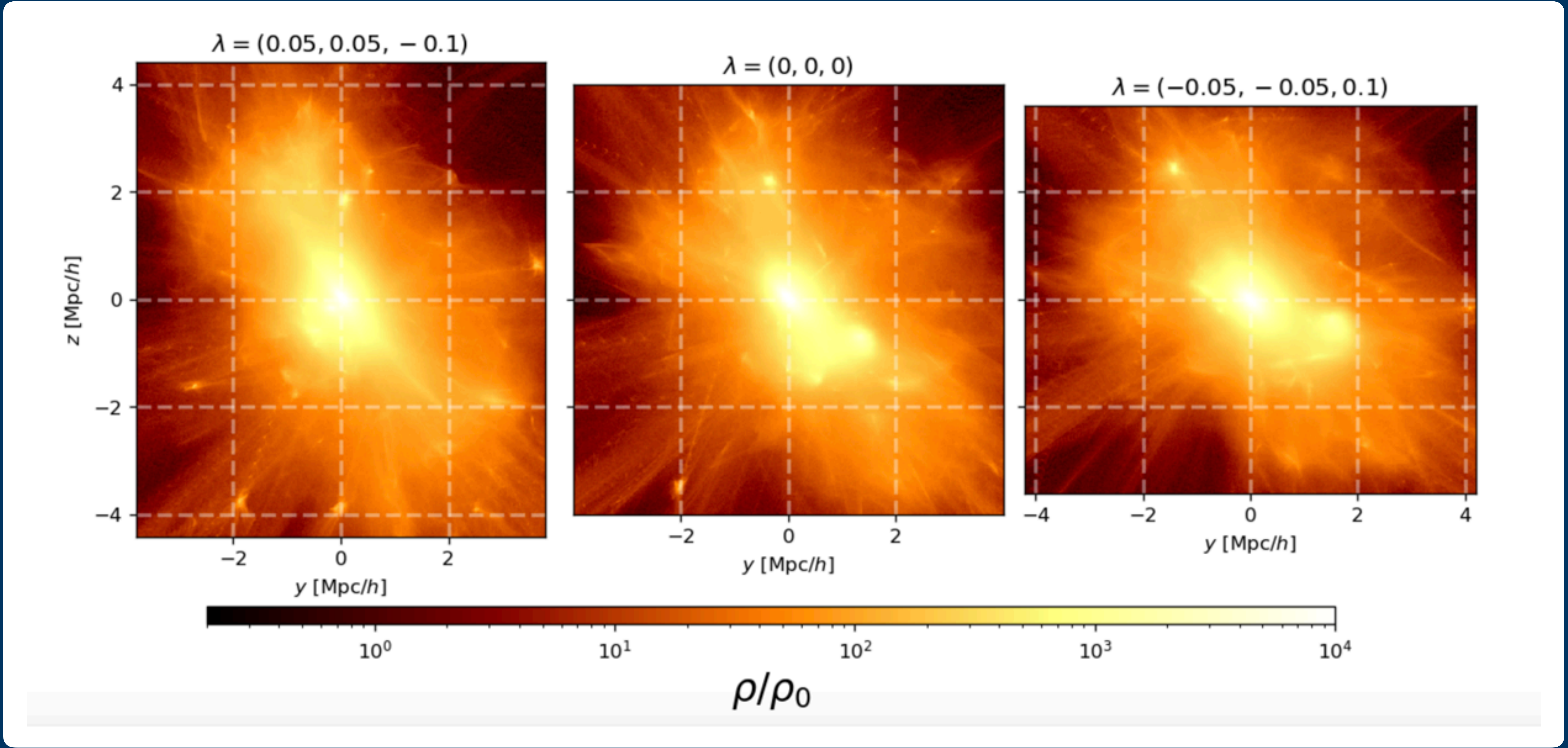


Jens Stucker



Raul Angulo

Probabilistic Shape Bias



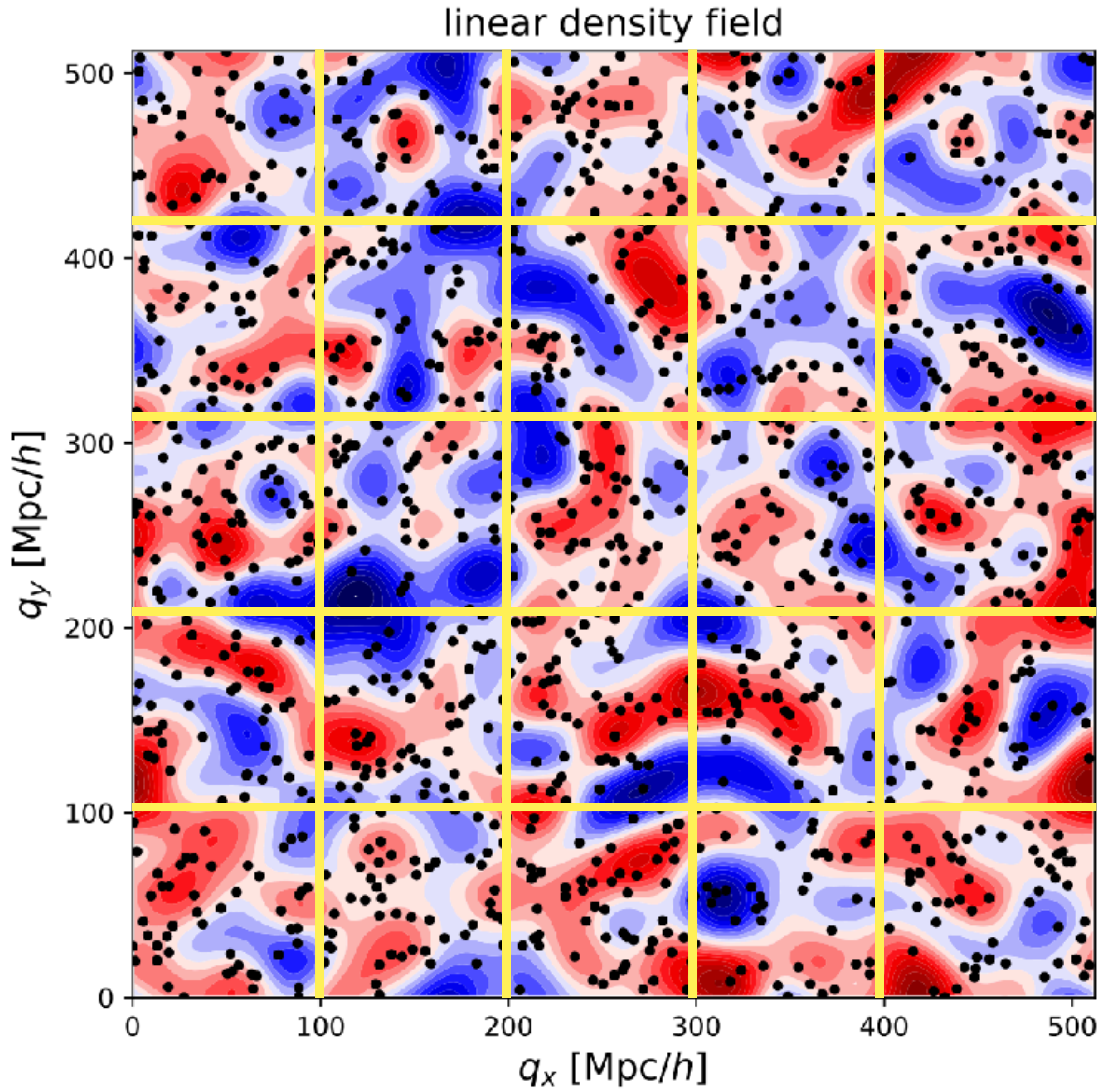
Stucker et al (2020)

Let I be the shape-tensor of halos/galaxies

$$\langle \mathbf{I} \,|\, \mathbf{T}_0 \rangle$$

$$\mathbf{C}_{K,n} = \frac{\partial^n \langle \mathbf{I} \,|\, \mathbf{T}_0 \rangle}{\partial \mathbf{T}_0^n} \Bigg|_{\mathbf{T}_0 = 0}$$

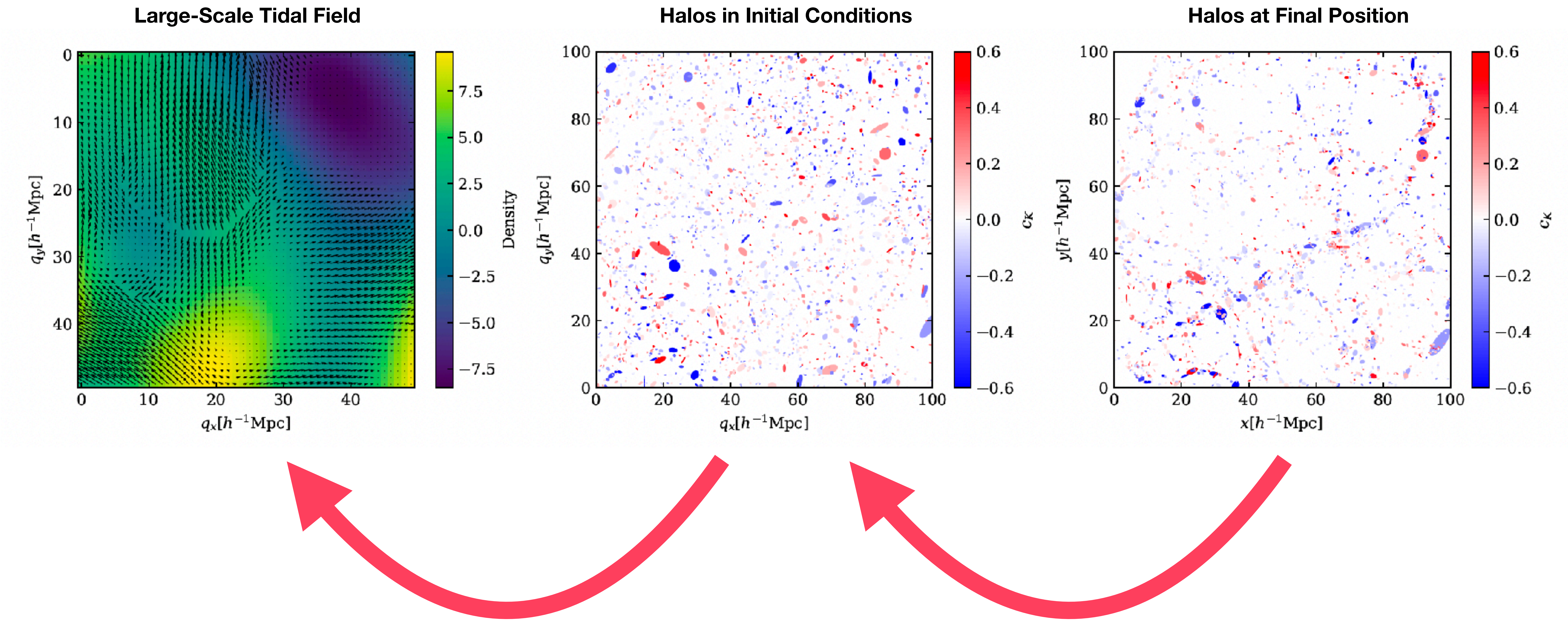
Probabilistic Shape Bias



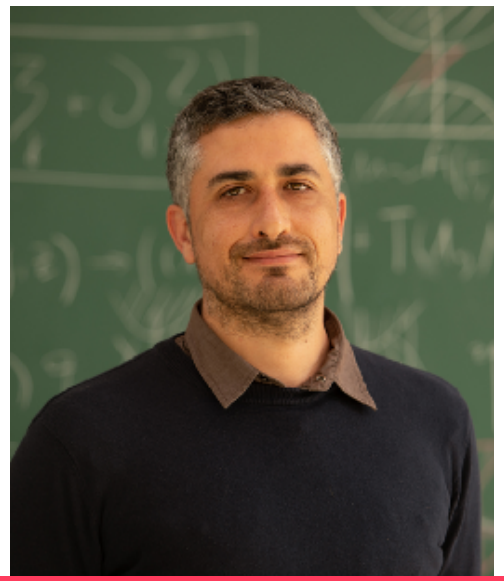
$$\langle \mathbf{I} \,|\, \mathbf{T}_0 \rangle_g = \frac{1}{F(\mathbf{T}_0)} \left\langle \mathbf{I} \frac{p(\mathbf{T} \,|\, \mathbf{T}_0)}{p(\mathbf{T})} \right\rangle_g$$

Stucker et al (2024)

Probabilistic Bias for IA

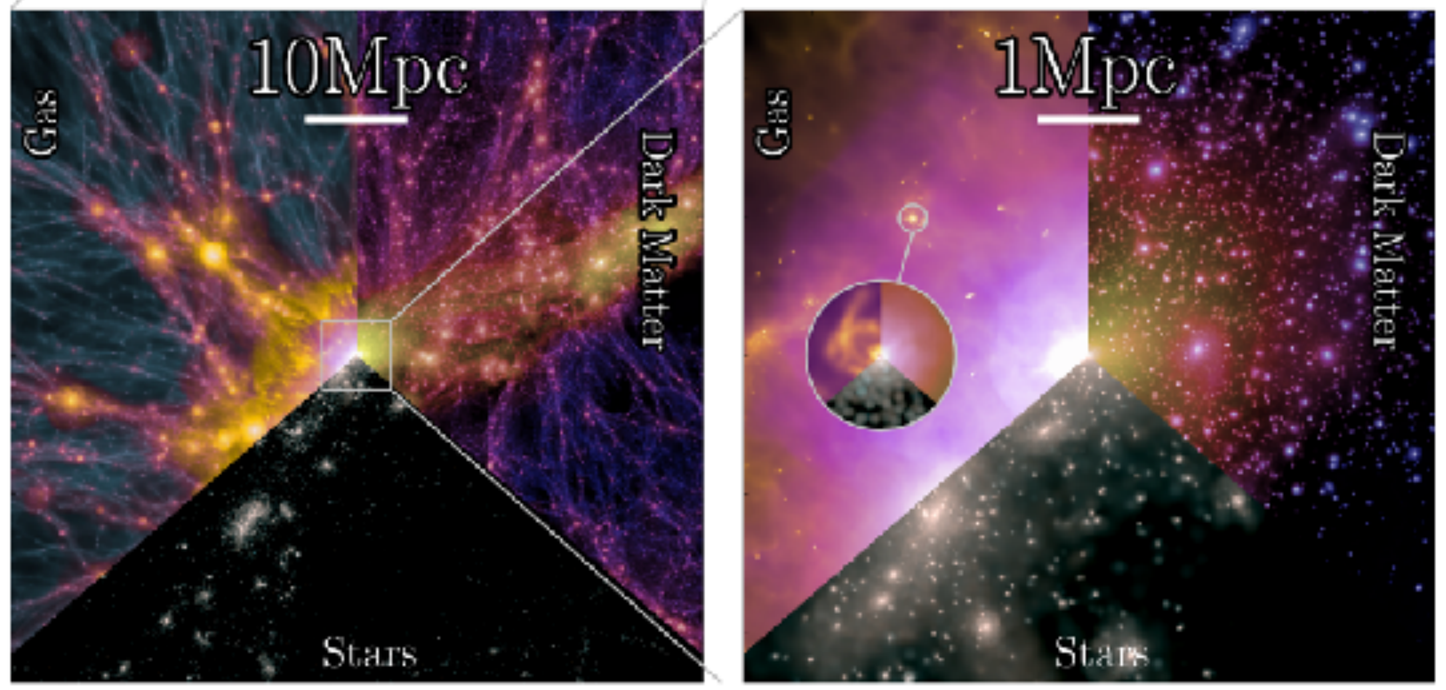


Probabilistic Bias for IA



+
MTNG Collaboration
+
many others

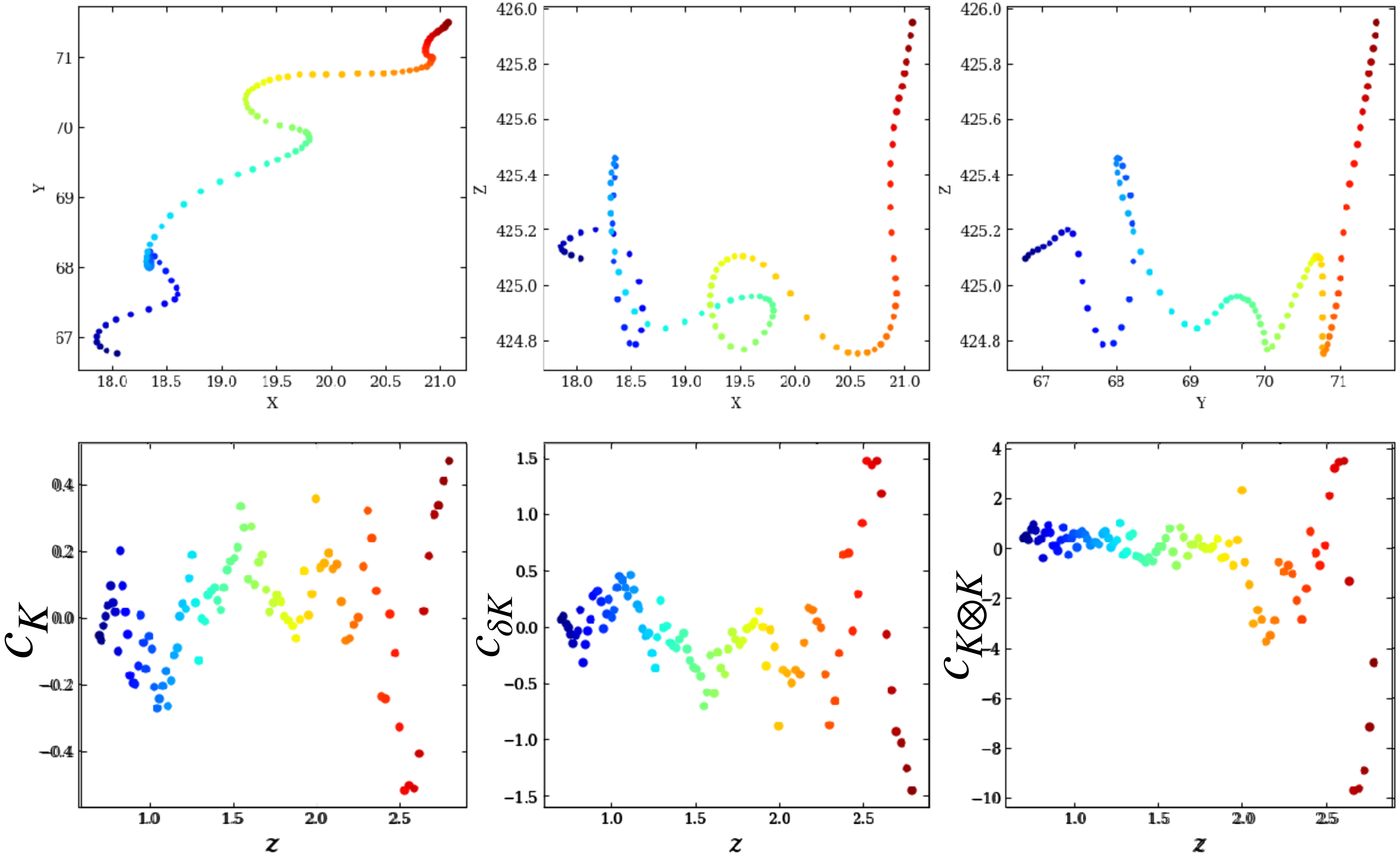
MillenniumTNG
Pakmor et. al (2022)

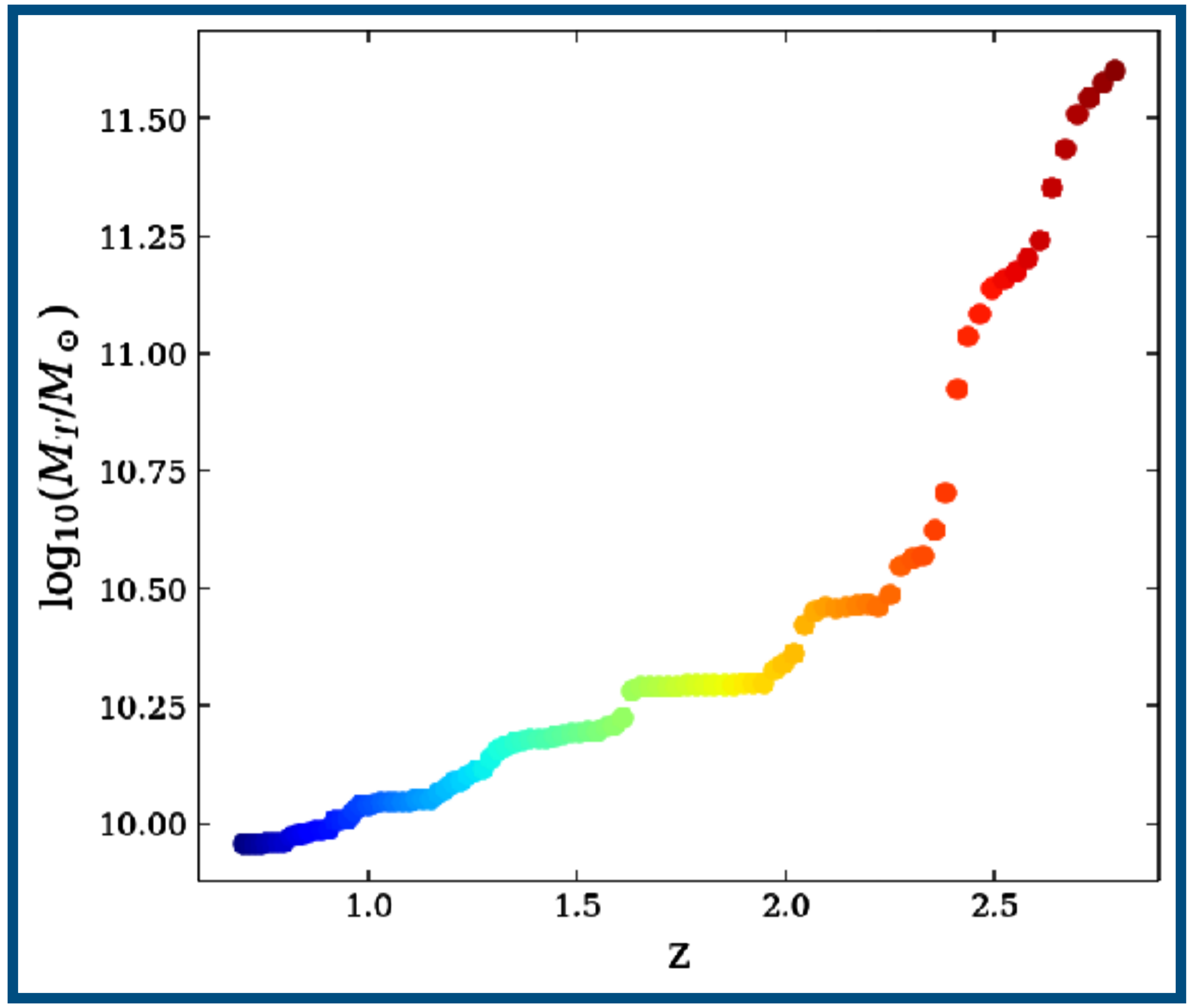


FM+(in prep)

Probabilistic Bias for IA

Tracing galaxies, their biases and mass throughout merger-trees



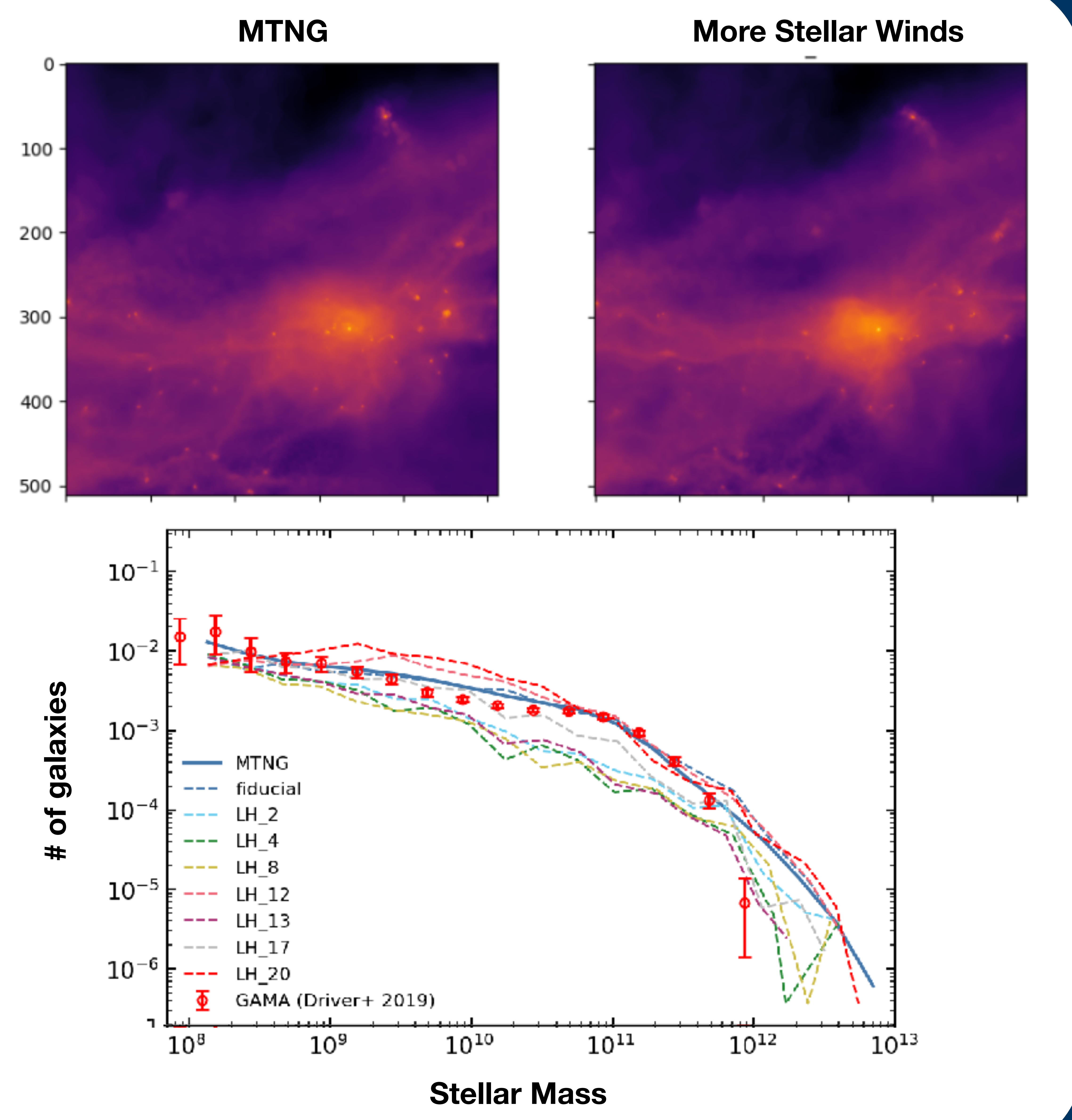


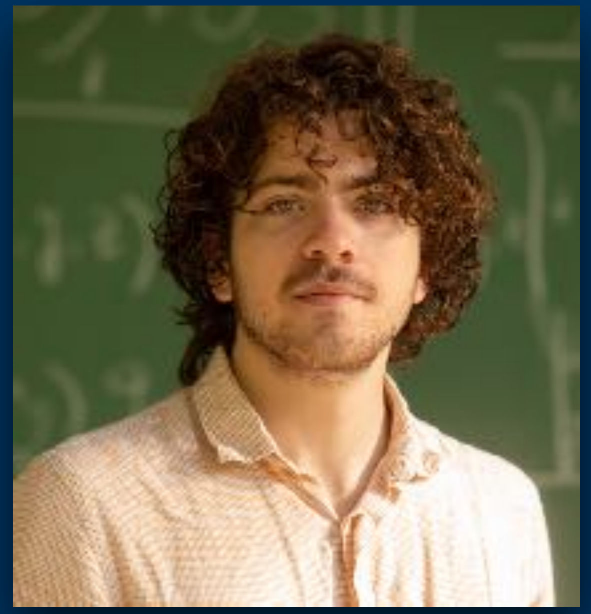
FM+(in prep)

Probabilistic Bias for IA

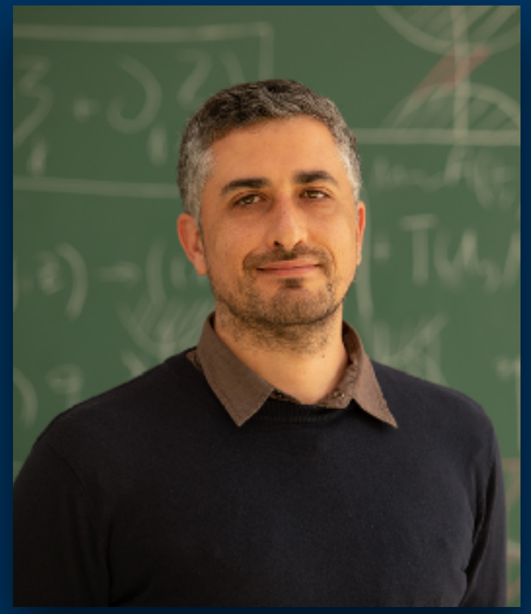
FM+(in prep)

MillenniumTNG





Francisco Maion
(Co-PI)



Raul Angulo
(Co-PI)

Volker Springel

+
MTNG Collaboration
+
many others

- ❖ Carefully selected set of 500 DM-halos
- ❖ Varying 7 parameters of the IllustrisTNG GFM
    - ❖ Stellar Winds
    - ❖ BH Feedback
    - ❖ Star-Formation Efficiency
- ❖ 30 points distributed in a wide Latin-Hypercube design
- ❖ 100k CPU-hours per resimulation

Conclusions

- IA modelling is crucial
  - ❖ Extracting info. from Euclid, LSST
  - ❖ Relevant from linear to non-linear regime
  - ❖ HYMALAIA goes well beyond linear regime
  - ❖ Precise with variance reduction
- Learning from simulations
  - ❖ Developed new estimators of shape bias
  - ❖ Priors from hydrodynamical simulations
  - ❖ Constrain shape-formation scenarios
- IA vs Baryonic Feedback
  - ❖ Innovative multi-zoom simulations with various sub-grid parameters

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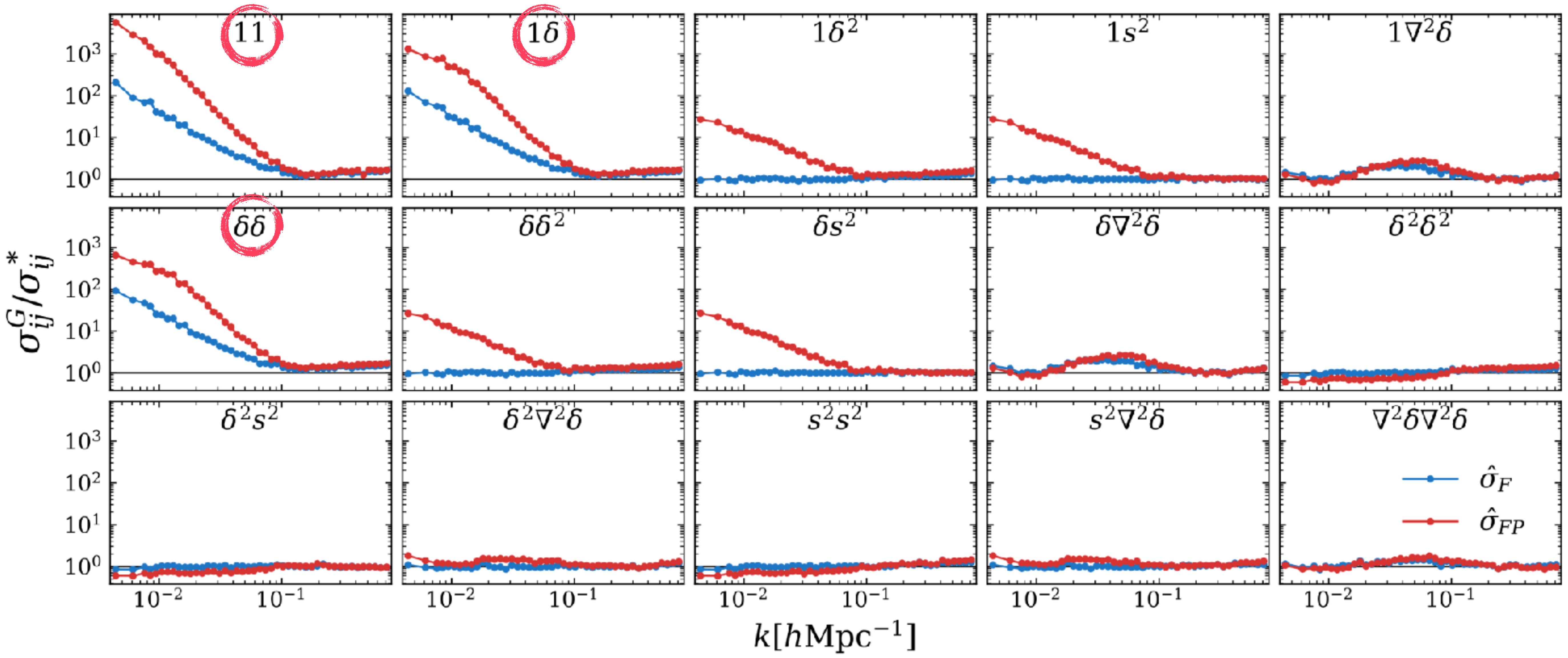
Extra Slides

Qualitative Understanding



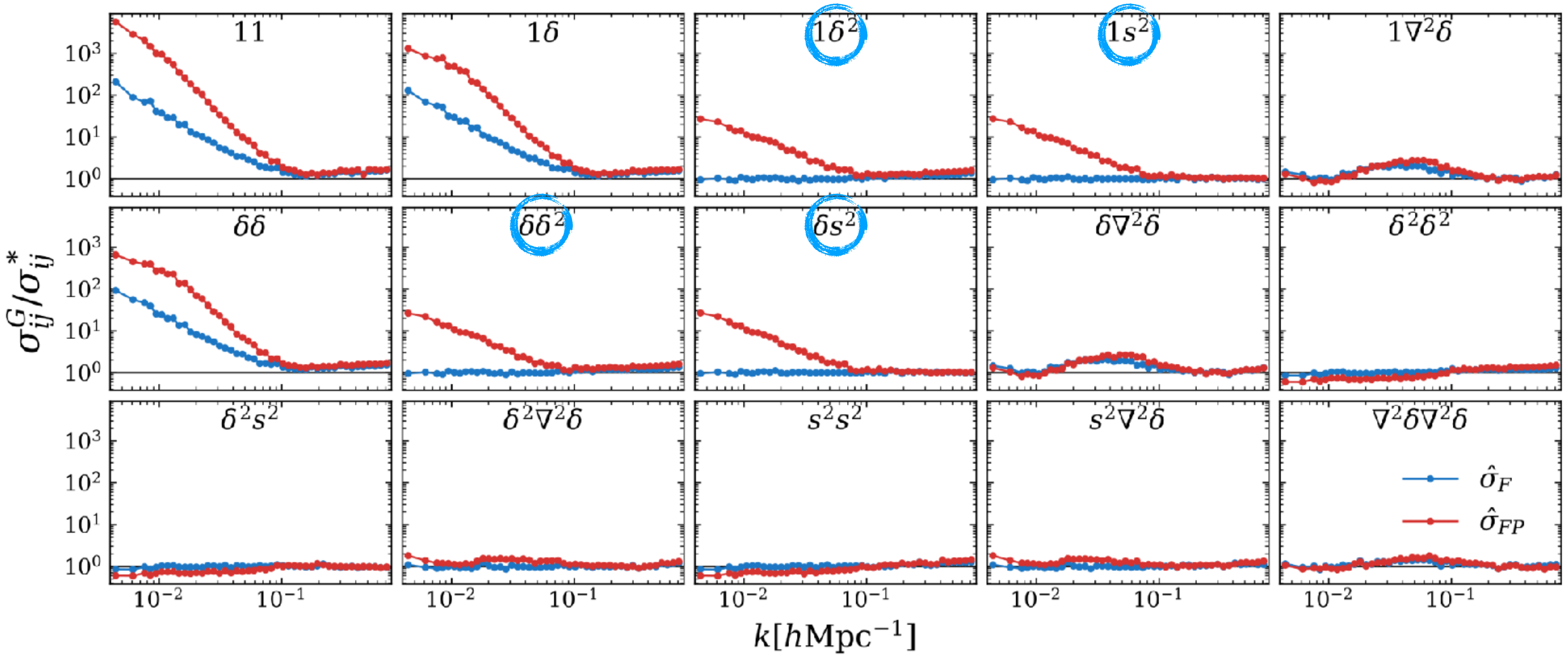
$$P_{11}, P_{1\delta}, P_{\delta\delta} \supset P^L$$

$$(\delta\delta\delta)_\pi \sim \int_{\boldsymbol{q}_1} \sqrt{\cdots}\, \underbrace{\cos\left[\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_1-\boldsymbol{k}} - \pi\right]}_{-\cos\left[\theta_{\boldsymbol{k}}-\theta_{\boldsymbol{q}_1}-\theta_{\boldsymbol{q}_1-\boldsymbol{k}}\right]} F_{ZA}(\boldsymbol{q}_1, \boldsymbol{k}-\boldsymbol{q}_1, \boldsymbol{k})$$
$$= -(\delta\delta\delta).$$

$$P_{11}, P_{1\delta}, P_{\delta\delta} \supset (\delta\delta\delta)$$

$$P^F_{11}(\boldsymbol{k}) \approx P^L_{\boldsymbol{k}} + V^{1/2} \int_{\boldsymbol{q}_1} \sqrt{P^L_{\boldsymbol{k}} P^L_{\boldsymbol{q}_1} P^L_{\boldsymbol{q}_1-\boldsymbol{k}}} \cos\left[\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_1-\boldsymbol{k}}\right] F_{ZA}(\boldsymbol{q}_1, \boldsymbol{k}-\boldsymbol{q}_1, \boldsymbol{k})$$
$$+ \frac{V}{4} \int_{\boldsymbol{q}_1} \int_{\boldsymbol{q}_2} \sqrt{P^L_{\boldsymbol{q}_1} P^L_{\boldsymbol{k}-\boldsymbol{q}_1} P^L_{\boldsymbol{q}_2} P^L_{\boldsymbol{q}_2-\boldsymbol{k}}} \cos\left[\theta_{\boldsymbol{q}_1} + \theta_{\boldsymbol{k}-\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_2} - \theta_{\boldsymbol{k}-\boldsymbol{q}_2}\right]$$
$$\times F_{ZA}(\boldsymbol{q}_1, \boldsymbol{k}-\boldsymbol{q}_1, \boldsymbol{k}) F_{ZA}(\boldsymbol{q}_2, \boldsymbol{k}-\boldsymbol{q}_2, \boldsymbol{k}).$$

Qualitative Understanding



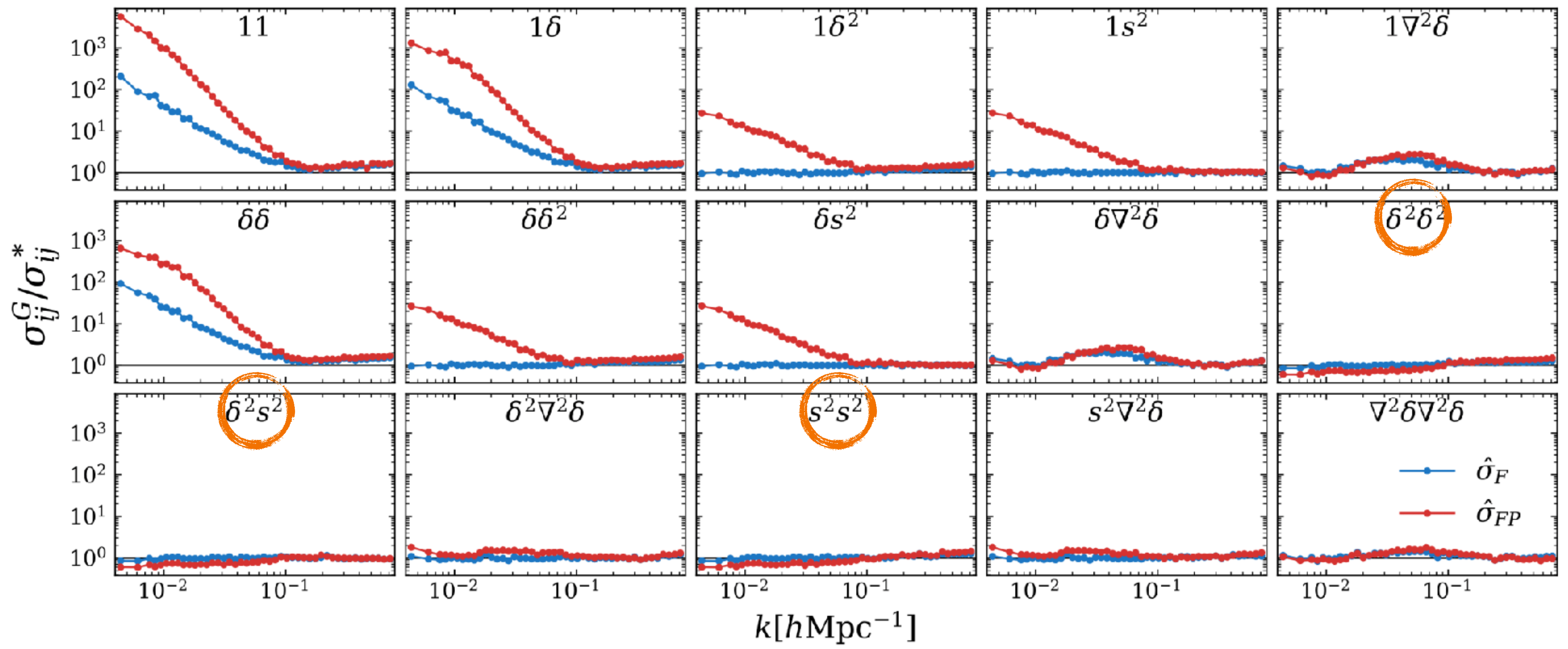
$$(\delta\delta\delta)_\pi \sim \int_{\boldsymbol{q}_1} \sqrt{\cdots}\,\underbrace{\cos\left[\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_1-\boldsymbol{k}} - \pi\right]}_{-\cos\left[\theta_{\boldsymbol{k}}-\theta_{\boldsymbol{q}_1}-\theta_{\boldsymbol{q}_1-\boldsymbol{k}}\right]} F_{ZA}(\boldsymbol{q}_1, \boldsymbol{k}-\boldsymbol{q}_1, \boldsymbol{k})$$

$$= -(\delta\delta\delta).$$

$$P_{11}, P_{1\delta}, P_{\delta\delta} \supset (\delta\delta\delta)$$

$$P^F_{1\delta^2}(\boldsymbol{k}) \approx V^{1/2}\int_{\boldsymbol{q}_1}\sqrt{P^L_{\boldsymbol{k}}P^L_{\boldsymbol{q}_1}P^L_{\boldsymbol{q}_1-\boldsymbol{k}}}\cos\left[\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{q}_1} - \theta_{\boldsymbol{k}-\boldsymbol{q}_1}\right]$$
$$+ V\int_{\boldsymbol{q}_1}\int_{\boldsymbol{q}_2}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q}_{12})}{|\boldsymbol{k}-\boldsymbol{q}_{12}|^2}\sqrt{P^L_{\boldsymbol{k}}P^L_{\boldsymbol{q}_1}P^L_{\boldsymbol{q}_2}P^L_{\boldsymbol{k}-\boldsymbol{q}_{12}}}\cos\left[\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_2} - \theta_{\boldsymbol{k}-\boldsymbol{q}_{12}}\right]$$
$$+ \frac{V}{2}\int_{\boldsymbol{q}_1}\int_{\boldsymbol{q}_2}\mathcal{K}_1(\boldsymbol{q}_1, \boldsymbol{k}-\boldsymbol{q}_1)\sqrt{P^L_{\boldsymbol{q}_1}P^L_{\boldsymbol{k}-\boldsymbol{q}_1}P^L_{\boldsymbol{q}_2}P^L_{\boldsymbol{k}-\boldsymbol{q}_2}}\cos\left[\theta_{\boldsymbol{q}_1} + \theta_{\boldsymbol{k}-\boldsymbol{q}_1} - \theta_{\boldsymbol{q}_2} - \theta_{\boldsymbol{k}-\boldsymbol{q}_2}\right]$$

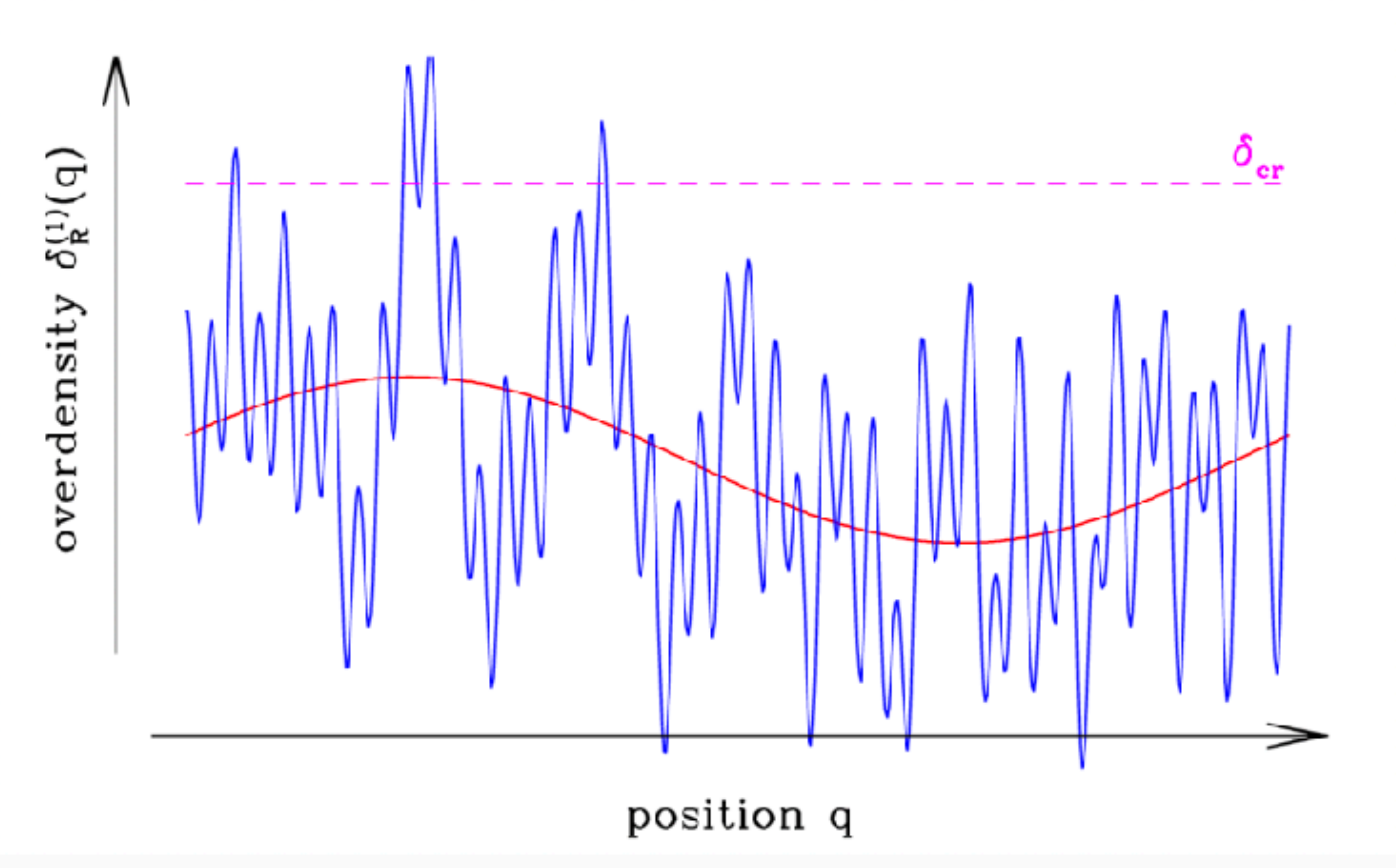
Qualitative Understanding



$$P^{F\&P}_{\delta^2\delta^2} \approx \frac{1}{V_f}\int_{\boldsymbol{q}_1}\int_{\boldsymbol{q}_2}\sqrt{P^L_{\boldsymbol{q}_1}P^L_{\boldsymbol{k}-\boldsymbol{q}_1}P^L_{\boldsymbol{q}_2}P^L_{\boldsymbol{k}-\boldsymbol{q}_2}}\cos\left[\theta_{\boldsymbol{q}_1}+\theta_{\boldsymbol{k}-\boldsymbol{q}_1}-\theta_{\boldsymbol{q}_2}-\theta_{\boldsymbol{k}-\boldsymbol{q}_2}\right]$$

Probabilistic Bias for IA

PBS Formalism



Desjacques+2016

Let f be the local density bias function

$$f(\mathbf{T}) = \frac{p(g\,|\,\mathbf{T})}{p(g)}$$

and

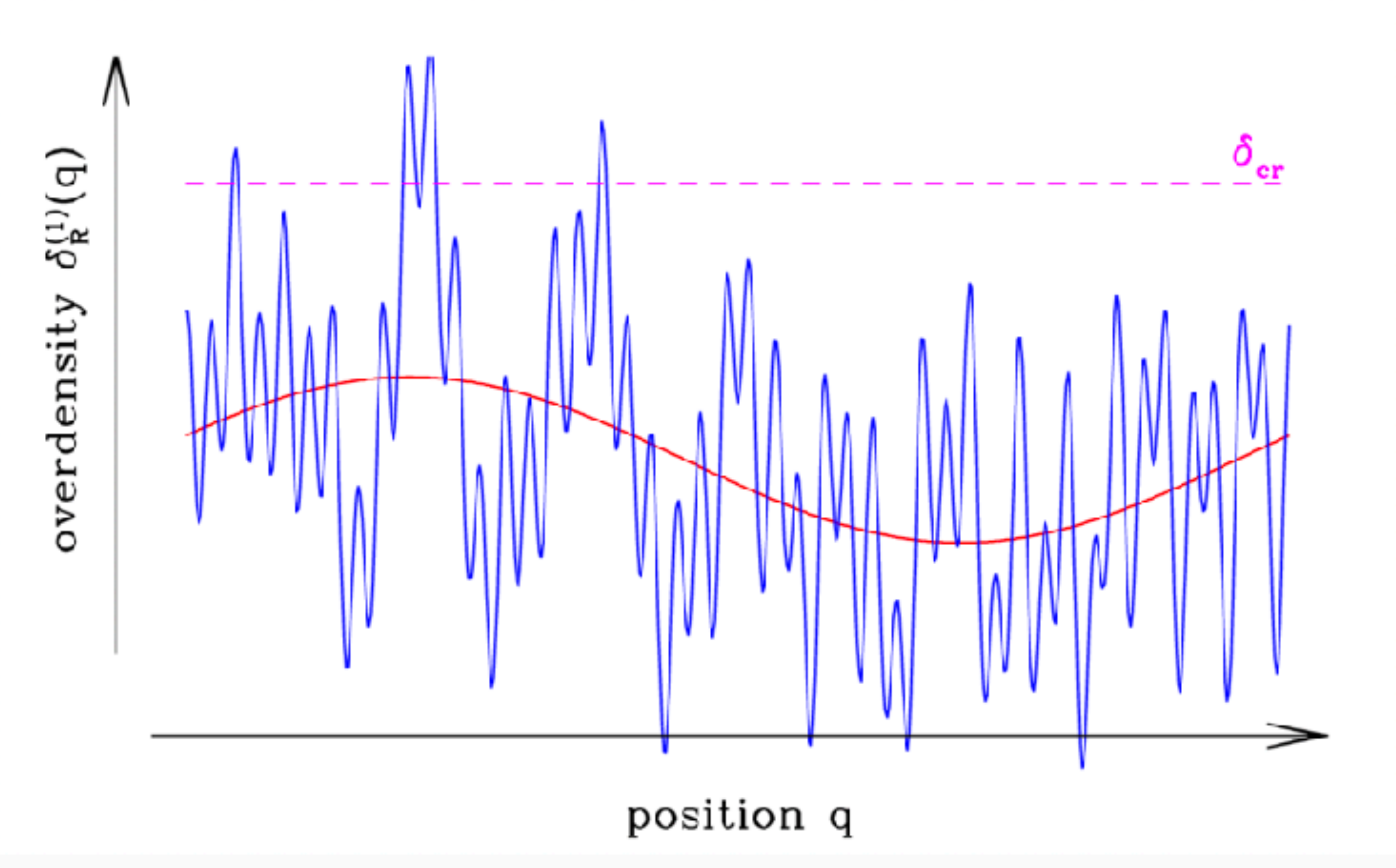
$$\langle h\,|\,x\rangle = \frac{\int \langle h\,|\,x,y\rangle p(y\,|\,x)\mathrm{d}y}{\int p(y\,|\,x)\mathrm{d}y}$$

then

$$\langle \mathbf{I}\,|\,\mathbf{T}_0\rangle = \frac{1}{f(\mathbf{T}_0)}\int \langle \mathbf{I}\,|\,\mathbf{T}_S\rangle_g f(\mathbf{T}_S) p(\mathbf{T}_S\,|\,\mathbf{T}_0) d\mathbf{T}_S$$

Probabilistic Bias for IA

PBS Formalism



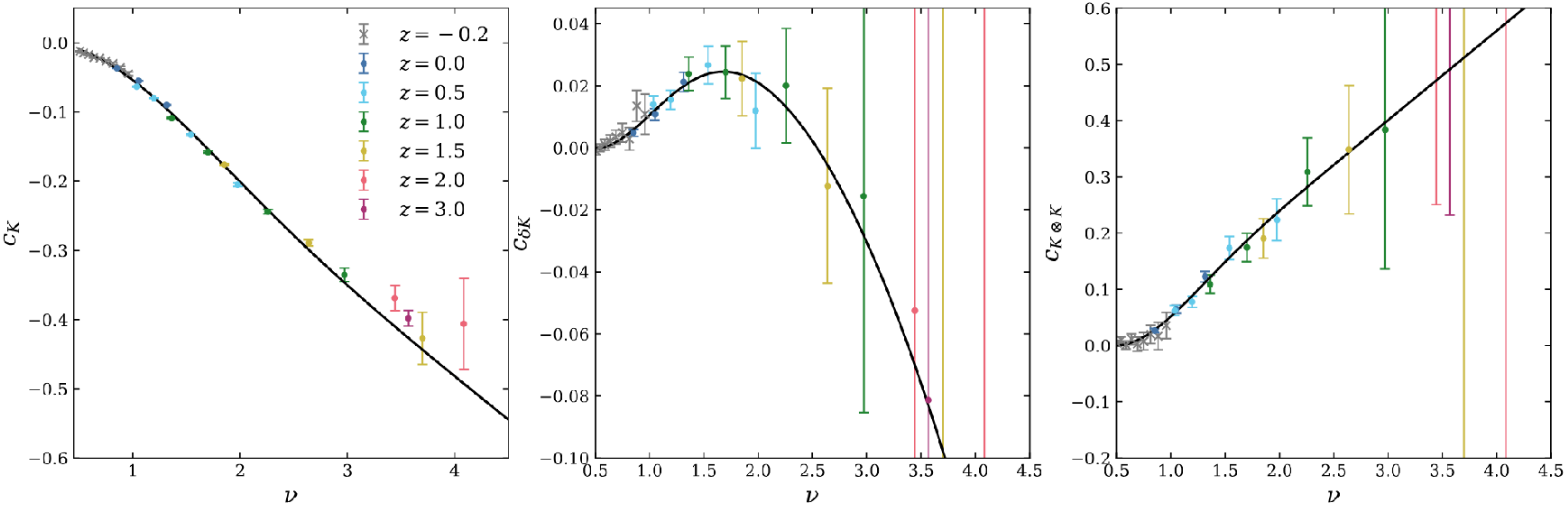
Desjacques+2016

Per-object Bias Estimators

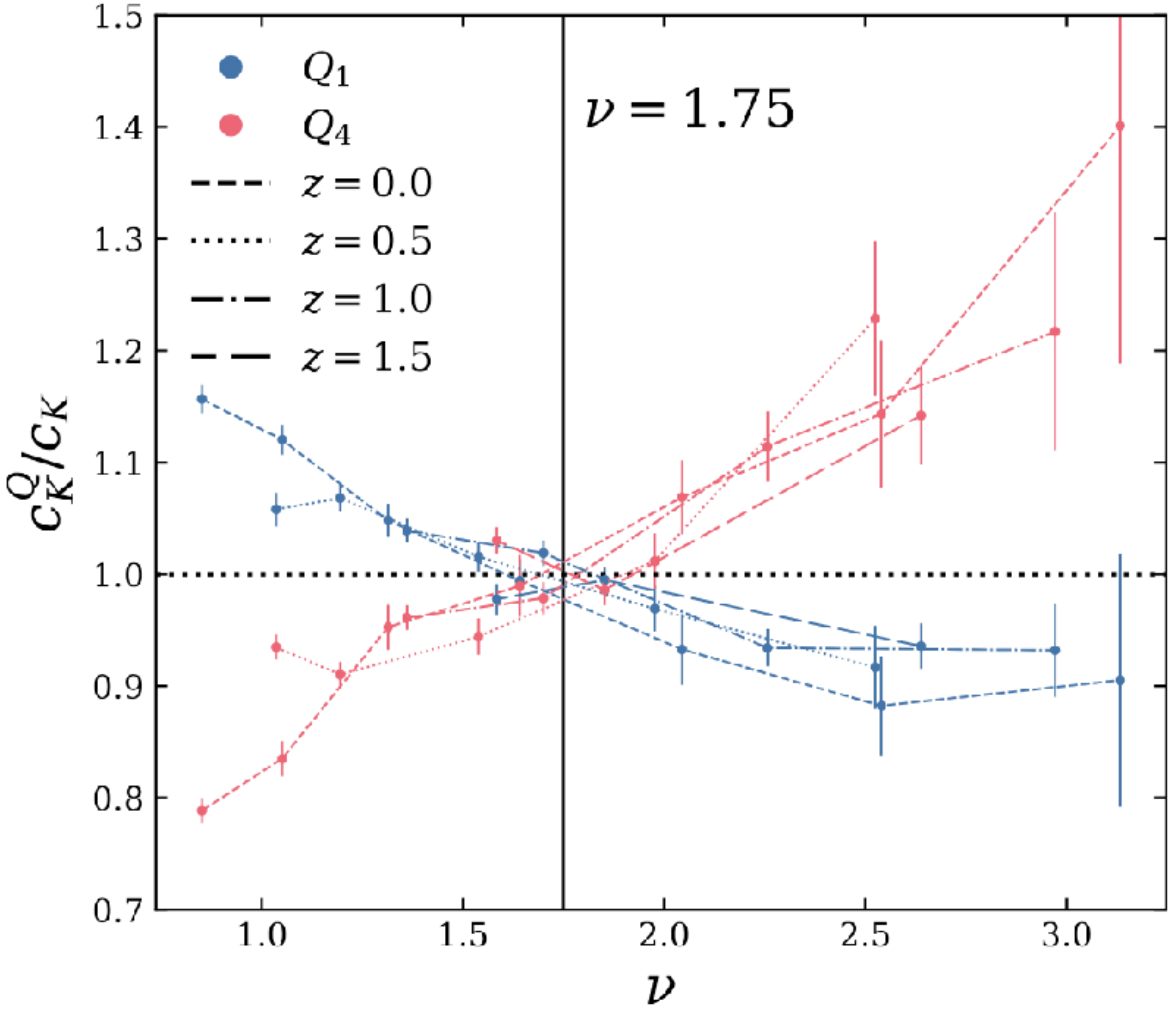
$$\mathbf{C}_{K,n} = \frac{\partial^n \langle \mathbf{I} \,|\, \mathbf{T}_0 \rangle}{\partial \mathbf{T}_0^n}\Big|_{\mathbf{T}_0=0}$$

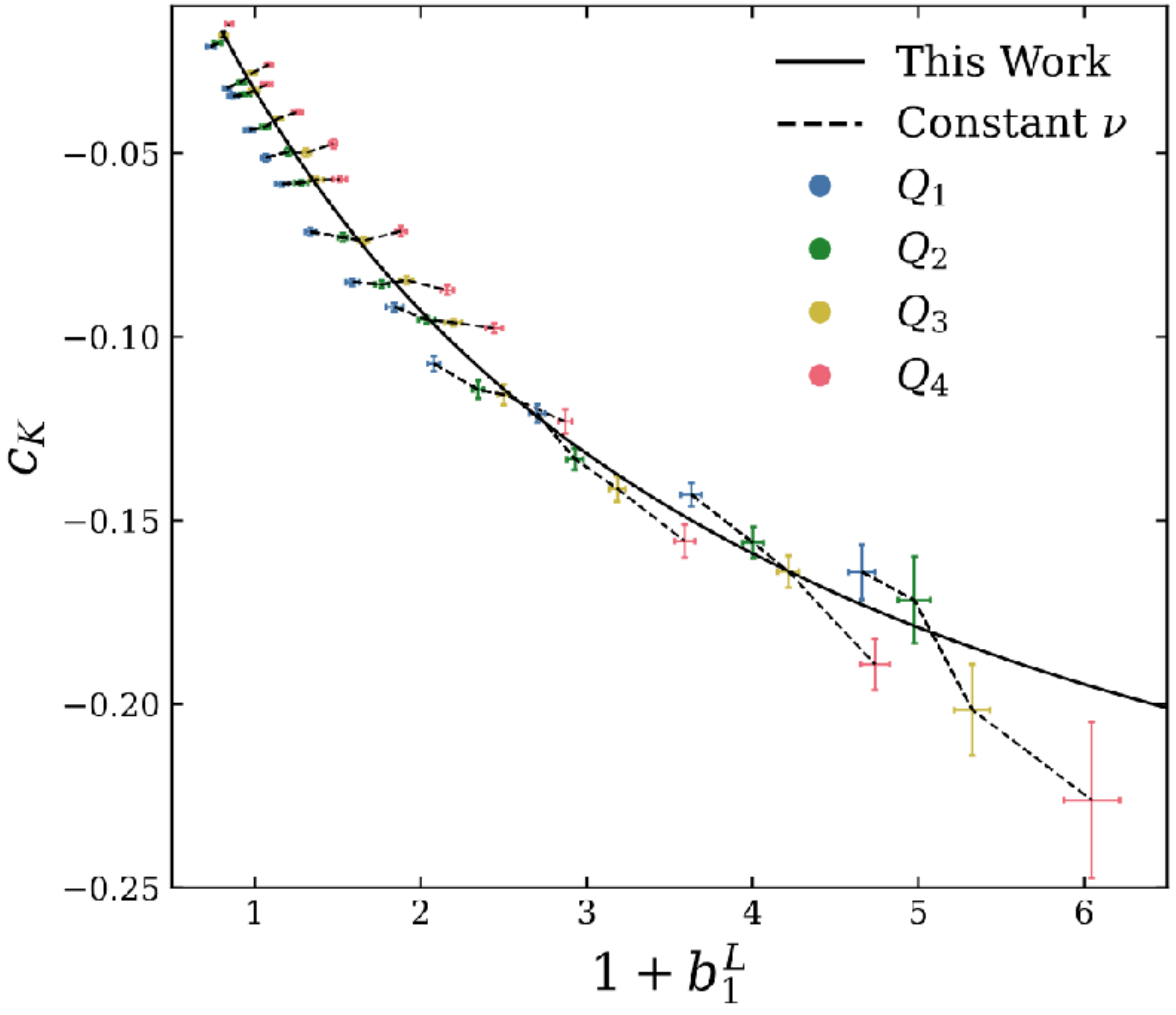
$$c_K = -\frac{3}{2}\mathrm{tr}(\mathbf{KI})$$

Universal Relation



Secondary Dependence





Linear Lagrangian Bias